The Evolution according to De Giorgi's Ideas

Marco Forti
Dip. di Matematica Applicata "U. Dini"
Università di Pisa, Italy
forti@dma.unipi.it

ECMTB2002 *"Mathematical Modeling & Computing in Biology and Medicine"*
5^{th} Conference of the European Society of Mathematical and Theoretical Biology

Minisymposium "Axiomatic approach to the evolution theories"

Milano, 2 luglio 2002.

De Giorgi's foundational programme was developed around his weekly seminar at the Scuola Normale Superiore in Pisa.

Foundations were not intended to give *safe and unquestionable grounds* to scientific theories, but rather to provide *conceptual environments* where research and criticism, in all sciences, could be carried out rigorously, but without the artificial constraints deriving from exasperate reductionism and formalization.

The premature death of Ennio De Giorgi slowed down the development of the programme, which is hopefully still alive.

De Giorgi's Main Principles

can be epitomized by the words of Hamlet:

There are more things in heaven and earth, . . . , than are dreamt of in your philosophy.

- *Non-reductionism*
- *Open-endedness*
- *Self-description*
- *Semiformal approach*

Non-reductionism

There are many kinds of *qualitatively* different objects and concepts which are studied in the Sciences and Humanities. The mathematical modelling of many scientific concepts has been fruitful in providing a *quantitative analysis* of these concepts. But a total reduction of *natural scientific notions* to their mathematical codings can undermine the conceptual clarity of the notions themselves. Instead, the possibility of using *qualitatively*, and not only *quantitatively* different kinds of objects simplifies the enterprise of introducing new notions, especially in the Sciences of Nature.

For instance, *biological objects* are better defined *per se* and not on the basis of a decomposition into simpler constituents, whose mere sum might not measure up to an adequate definition of the biological object itself.

An abstract concept, like that of *species* cannot be adequately accounted for as a *set of individuals*, since it has properties, and it partakes in relations, as a *single complex* entity, independently of the specific set of individuals present at any given time. A scientific study of a species, in effect, has to consider its interactions with the *whole* ecosystem, and also its historical development.

Even in pure Mathematics, where the twentieth century's *set-theoretic reductionism* has obtained very important results, a "totalitary" interpretation may render it difficult, even impossible at times, to formulate appropriate axioms and conjectures.

- The intuitive notion of *operation* subsumes the *intensional* concept of computation process, and so operations cannot be simply coded by their graphs, or reduced to input-output diagrams.

- Taking *natural numbers* as primitives allows for a clearer analysis of the connections between their different implementations, not only as Von Neumann ordinals, but also as Frege-Russell cardinals, Church numerals, etc.

- Conceiving *collections* as *truth-valued operations* would force unnecessary commitments on the definition of collection, and yet it would not make apparent their intrinsic *extensionality*.

General non-reductionist axiomatic theories can appear *prima facie* unnecessarily complicated, but this drawback seems unescapable, if we want to represent *naturally* within the framework, at least in principle, the multitude of existing conceptual systems.

Open-endedness

An axiomatic framework should be open to extensions in any conceivable direction. The natural introduction of any sufficiently analyzed and clarified concept should be always possible. In axiomatizing a specific discipline, primitive notions and axioms should be formulated so as to allow for the introduction of *new objects* and *concepts* which can emerge from theoretical and experimental developments of the discipline. And this goal should possibly be achieved in a *conservative* way, without having to revise the formulation of the main properties previously axiomatized.

In particular, when axiomatizing a general foundational theory, one should look for a framework suitable for accommodating most of classical and modern theories arising in Mathematics, Logic, Computer Science, as well as in Physics, Biology, Economics, Linguistics, etc.

And the axiomatization of any particular science has to take into account the whole range of its theoretical and experimental achievements. As a matter of fact, it is often difficult, even impossible at times, to integrate the huge quantity of new information within the current conceptual framework of the theory.

Self-description

A crucial instance for a foundational framework is the possibility of *self-description*:

- the most relevant operations and relations which the framework utilizes should themselves be first class objects in the framework.

For instance, various *qualities, relations* and *operations* should be introduced in order to classify and describe the behaviour of the different *kinds* of objects, including qualities, relations, operations themselves.

Semiformal approach

Foundations, like any scientific discipline, can be expounded in a rigorous, yet informal style, and investigated using the axiomatic method of traditional Mathematics. This seems of particular importance in most applied sciences, because these have not yet developed formal languages of their own.

In order to allow for critical analysis by scholars from different fields of research, a *rigorous* presentation using *natural language*, appears to be preferable to a *formal* axiomatization, carried out within some *artificial* language from Mathematical Logic. Of course, one has to render the statements as unambiguous as possible, by giving a precise axiomatic determination of the basic technical terms.

Also in Mathematics, where formal languages are well developed and formal axiomatizations in First Order Logic have become customary, it turns out that *none* of them can be satisfactorily taken as definitive. In fact, any formalization misses part of the intended meaning of the original theory.

The above considerations are not intended to sound as a *depreciation* of the relevance formal theories have in the development of Sciences. De Giorgi termed his approach *semi*formal rather than *in*formal, precisely because he thougut that any interesting axiomatization should admit *easy, natural* translations in suitable formal languages, thus benefiting from various important results of modern Mathematical Logic.

More importantly, the *formalizations* of the axiomatizations of different disciplines can exhibit *formal* (structural) similarities and relations, and fruitfully suggest *substantial* analogies and connections between the original notions, not apparent before.

The axiomatic method

Any scientific discipline has a (more or less precisely determined) domain of study, which it intends to investigate. The output of this investigation amounts to a partially organized collection of important properties and facts pertaining to the objects of this domain. Axiomatization should be the very effective methodology for *discovering*, *organizing*, *analysing*, *criticizing* and *communicating* the aforementioned body of theoretical and experimental data.

The axiomatic method has two kinds of purposes and benefits, which clearly overlap in many instances:

- the internal, *intradisciplinary*, ones, which are functional to the *development of the discipline* itself,

- the external, *interdisciplinary*, ones, which concern the *communication across different scientific communities*.

Intradisciplinary functions

An effective axiomatization of a discipline starts by identifying some *basic* (*primitive*) notions and by stating as axioms a *few, simple, clear* judgements concerning these. This, somewhat arbitrary, choice is motivated by the possibility of deducing as logical consequences many previously known facts, as well as new interesting facts. This allows for a deeper conceptual organization of the existing knowledge and a widening of the theory.

The requirement of producing a preliminary list of the basic primitive notions of a theory is a very reasonable request, albeit not so easy to achieve in general. On the other hand the refusal of producing it might hide *ambiguities* and *contradictions* and the unwillingness to face them explicitly.

The axiomatization process can help in detecting also ambiguities which one was unaware of in the original conception of the discipline. In fact it allows to achieve a rigorous account of what is assumed and is implied, preventing from smuggling in unconsciously *intentions* and *connotations* which are not explicitly mentioned.

De Giorgi thought that even what apparently may seem the most undesirable outcome of an axiomatization, i.e. *the derivation of a contradiction*, has in fact an invaluable importance for the understanding of the principles of the theory, and sometimes can have a revoutionary impact on the discipline itself. Such a contradiction is the evidence that some *misconception* was hidden in our intuition of the very basic notions of the discipline.

Cases in point are

- Frege's axiomatization of the notion of *class*,
- Linnæus' fixist theory of *species*.

  [this latter example is indirect, since Linnæus did not formulate an explicit axiomatic theory.]

Interdisciplinary functions

The fact that an axiomatization of a given field is based on *few, primitive, clearly isolated notions*, and *few, simple, concise statements*, helps in furthering the dialogue between scholars of diverse disciplines. Hence the axiomatic method is a medium for the fostering of constucitve criticism and it allows for the importation into the discipline of original suggestions stemming from even the most remote experiences. Moreover a rigorous axiomatic presentation reduces the possibility of those misunderstandings which often occur in interdisciplinary communication.

The important fact that, in principle, the axioms *contain already all* their logical consequences, can facilitate the difficult task of obtaining the conciseness necessary for a successful dialogue.

The axiomatization process provides the *most objective* formulation of a theory and this makes interested scholars less likely to be influenced by the authority of the authors or by their own prejudices upon them.

Sapiential aspects of axiomatization

De Giorgi maintained that a deep understanding of the foundational problems could unveil the *sapiential aspects* of the scientific activity. The conceptual clarity and the semiformal rigour of the traditional axiomatic method is incomparable in facilitating critical contributions of scholars from different fields. Therefore the foundational programme fitted perfectly with De Giorgi's ethical conception of scientific research. Open and sincere dialogue among scholars of different attitudes, and conviviality in sharing of knowledge are the main factors of comprehension, friendship and mutual respect.

Although De Giorgi has been always ready to put himself into the struggle in defense of the fundamental human rights, nevertheless in his opinion theoretical analysis was much more relevant also in this respect. Illuminating of this attitude is his repeated affirmation that one of the most important axiomatic systems, one worth being studied in any order of schools, is the UDHR, the Universal Declaration of Human Rights of 10/12/1948.

Limits of the axiomatic method

Any axiomatization gives only a *partial* account of the domain of investigation. This is rather clear for experimental Sciences, where only a small part of the interesting phenomena has a counterpart in the theory. But even in elementary Arithmetic we know, after Gödel, that

- *no effective axiomatization* can derive all truths involving sums and products of natural numbers.

Thus also in this case one has always to search for *new powerful* axioms, which can enhance the deductive strength of the theory.

One should always consider simultaneously *various different possible axiomatic frameworks* for any single discipline, whose various aspects and facts are often difficult to organize in a unified framework, based on few previously isolated primitive notions. A critical analysis and comparison of these systems may lead to a new synthesis, possibly based on new primitives, accomodating simultaneously most of the relevant data under consideration.

The axiomatic systems of 2000

During the last years of De Giorgi's life, the scope of his foundational programme widened, and centred on the formulation of general axiomatic frameworks suitable for expressing (*engrafting*) Mathematics and Logic, Informatics, Biology, Economics and, possibly, any sufficiently clear conceptual domain arising in the Sciences and Humanities.

To this aim, only the following fundamental notions are isolated:

- *qualities, relations, operations, collections,* and *sets.*

The primitive notions

Most disciplines in the Sciences and Humanities deal, ultimately, with *several qualitatively different* objects and study *properties, relations* and *operations* over them. Therefore, in order to design such a general axiomatic framework, it seems appropriate to select the *pre-mathematical* notions of *quality* (or *property)* and *relation* assuming only them as *primitive*, i.e. not reducible to preceding concepts.

Most axiomatizations of scientific theories make use of two other primitive kinds of objects: *operations* and *collections* (and sets). According to the basic pattern for engrafting new concepts in the general axiomatic framework, we introduce these notions by means of suitable *qualities*, which classify the objects under consideration, and of suitable *relations*, which describe their behaviour and connect them with the objects introduced previously. Thereafter, also operations and collections will be used together with the other basic notions for axiomatizing new theories.

A serious problem arises in dealing with *experimental sciences*, and especially Biology, within a foundational framework *à la* De Giorgi: dew to its origin in Mathematics, Logic and Computer Science, this framework accommodates naturally only abstract, "eternally invariant" objects. A few attemps have been made of engrafting "variation", at the prize of some (perhaps cumbersome) parametrizations. We refer only to the introduction of the concept of *variable* of classical Mathematical Physics suggested in [Var94], and to the treatment of *contingency* and *modalities* outlined in [ont99]. However much more is needed for superseding the obstacles met in axiomatizing biological notions, in particular when dealing with *living objects*: for instance, a lion is a living object until it dies, it is in the relation $Rcomp$ with its stomach until it is removed, etc.

To be sure, the very primitive notion of biological object seems to bring about a kind of dependence on *time* and *space* parameters, thus leading to a general notion of *variable object*.

We start from the naive idea that the objects of natural sciences (say the $Earth$, or the famous policedog Rex) have an *individual* nature wich goes beyond the *singular concrete things* with which they are identified at each instant (usually a suitable finite aggregate of particles). Of course, depending on the objects under consideration, not only *time* instants or intervals come into play, but also *spatial* determinations and more generally any kind of relevant data suitable for characterizing a given situation. Therefore we introduce a very large and unspecified collection $\mathcal{S}$ of parameters or "sorts" (the *states* of the universe), conceived to encompass all "phase spaces" considered by any scientific theory under consideration, in particular here Biology. At suitable values of the parameters, any natural object has a "concrete representative" (the *Earth* is an aggregate of determined molecules, and *Rex* of determined cells). Depending on the state, these concrete representatives *change* (some atoms of the Earth are eliminated or substituted, some old cells disappear in generating new ones inside Rex, etc.), while the "abstact object" is still considered to remain the *same*.

The *variable objects*

We introduce

Qvar, the quality of being a *variable* object

$\mathcal{S}$, the collection of the *sorts* or *states*

Ract, the relation of *actualization*

AXIOM 1 *$Qvar$ is a quality, $\mathcal{S}$ is a collection, and $Ract$ is a ternary relation s.t.*

$$Ract\ x, y, z \implies Qvar\ x \ \wedge\ y \in \mathcal{S};$$

$$Ract\ x, s, y \ \wedge\ Ract\ x, s, z \implies y = z.$$

If X is a variable object, $s \in \mathcal{S}$ is a sort, and $Ract\ X, s, z$, then z is denoted by X^s and called the *actualization of X at the state s*. If X has an actualization X^s at the state s we say that X *exists at* s, and that X^s *is* X *at* s.

One has to investigate the connections between a variable object and its instantiations, and the structure of the "phase space" $\mathcal{S}$ (as a matter of fact, this topic is still largely unexplored). We consider only one example, typical in Biology: the case of an operation acting on a variable object and producing another variable object. It seems convenient to assume that any such (abstract) operation can be "decomposed" into several (concrete) operations which, at suitable states, act on suitable (instantiations of) objects, and produce other (instantiations of) objects, possibly in new states. The first object has to be (a component of) the argument and the last one (a component of) the result of the abstract operation.

Operations on variable objects

AXIOM 2 *If F is an operation and X, Y are variable objects such that $FX = Y$, then there exist*

- *operations $f_1, \ldots, f_n$,*
- *variable objects $V^1, \ldots, V^n$, and*
- *states $s_1, \ldots, s_n \in \mathcal{S}$ such that*

$$f_i V^i_{s_i} = V^{i+1}_{s_i} \quad for \quad i = 1, \ldots, n-1.$$

Moreover $Rcomp\ X, V^1$ and $Rcomp\ Y, V^n$.

The operations $f_1, \ldots, f_n$, are called the *components* of the variable operation F.

We have considered only simple operations for sake of brevity, but clearly we intend to postulate a correspondent axiom for binary operations as well. Moreover binary operations can actually appear as components of simple variable operations and *vice versa*. (Alternatively, we could consider binary operations as acting on *pairs* of objects.)

Similar axioms should be given also for relations, qualities, or collections involving variables objects. Finally it should be appropriate to specify suitable relations among the states involved in such axioms.

BIOLOGICAL OBJECTS

"Biology is the Science which studies Biological Objects"

According to the general principles of our nonreductionist foundational theory, we do not intend to *define* once for all the notion of biological object in terms of *more fundamental* notions; rather we try to find out some of the main features of biological objects and we state them in form of axioms about the quality $Qbiol$ of being a *biological object*. We start by noticing that any biological object is involved in many fundamental relations: it has been *generated* by something, it *exchanges energy* with something, it is *composed* by parts and is itself part (*component*) of complex entities, etc.

In particular we consider $Rgen$, the most general relation between *generator* and *generated*. The intended meaning of $Rgen\ x, y$ is that y has been generated by x.

AXIOM 3 *$Qbiol$ is a quality. An object x has the quality $Qbiol$ if and only if it is a biological object. $Rgen$ is a binary relation. If $Qbiol\ x$ then there exists y such that $Rgen\ x, y$.*

Here $Rgen$ is conceived as the *most general* relation of generation, hence no particular "process of generation" is specified. It may be that x is itself a biological object, but it is also possible that this is not the case (e.g. in photosynthesis CO_2 and H_2O generate sugar). Several specifications of this general relation can be considered in different contexts:e.g. the *operations of generation* instantiate this relation when *living objects* are concerned and the *operations of speciation* when species are.

LIVING OBJECTS

The *living objects* are in fact *essentially variable*. We do not try to define the living objects by means of previously introduced concepts. We introduce a quality $Qviv$ (Lat. *vivens*) which characterizes all and only living objects.

AXIOM 4 *$Qviv$ is the quality of being a living object. If $Qviv\ x$, then also $Qvar\ x$ and $Qbiol\ x$.*

Any living object is involved in the general relations $Rcomp$, Ren, and $Rgen$. We can be more specific about $Rgen$, and we consider as a basic characteristic of living objects the fact of being involved in *operations of generation*.

Operations of generation

We introduce a collection $Opgen$ of operations of generation, with the intended meaning that any $G \in Opgen$ is a *variable operation* which applies to one or two suitable living objects and produces a (possibly empty) set of living objects.

AXIOM 5 *Opgen is a collection of variable operations (the operations of generation):*

1. if X, Y are living objects, $G \in Opgen$, and GX (or GXY) esists, then it is a set of living objects;

2. if $GX = \{Z_1, ..., Z_n\}$ (or respectively $GXY = \{Z_1, ..., Z_n\}$) then $Rgen\ X, Z_i$ (and respectively also $Rgen\ Y, Z_i$).

We have taken a *set* of living objects as result of an operation of generation since usually a single such operation can produce more than one individual. Roughly speaking, any operation of generation corresponds to a "specific way" of reproduction. If $g_1, \ldots, g_n$ are the components of $G \in Opgen$, then the operations g_i represent the different concrete actions which, at suitable states, are made by one or two living objects in order to generate other living objects. Thus one could start an analysis of the various forms of reproduction by simply considering the structure of the different components of any operation of reproduction.

The concept of species

We introduce the different species as *qualities of living objects*, together with the quality $Qspec$ of being a species.

A first nontrivial axiom postulates that any two individuals of the same species always have a common ancestor of the same species. In order to allow for the process of speciation, the converse property (namely that the outspring is always of the same species of the parents) has to be postulated cautiously, so as to admit exceptions at suitable states.

AXIOM 6 *$Qspec$ is the quality of being a species. If $Qspec\ S$ holds, then S is a quality of living objects. Moreover S is a biological object, i.e. $Qbiol\ S$.*

Let X, Y be living objects of the species S; then:

1. there exist living objects $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$, Z, all having the quality S, such that $Rgen\ X_1, X$, $Rgen\ X_2, X_1, \ldots$, $Rgen\ Z, X_n$, $Rgen\ Z, Y_m$, $Rgen\ Y_m, Y_{m-1}, \ldots, Rgen\ Y_1, Y_2$, $Rgen\ Y_1, Y$;

2. there exists a collection of exceptional states $\mathcal{E}_S \subset \mathcal{S}$ such that, if G is an operation of generation, $GXY = \{Z_1, \ldots, Z_n\}$, and all the components of G act at states $s_i \notin \mathcal{E}_S$, then all the Z_i belong to the species S.

The collection $\mathcal{E}_S$ in clause 2 takes into account the exceptional states corresponding to an *operation of speciation*. If during the time interval corresponding to $\mathcal{E}_S$ an operation of speciation produces a new species S', then some (or possibly all) descendants of S-individuals generated after the end of $\mathcal{E}_S$ will belong to the new species S'.

The contrast between the theory of evolution and the concept of species, born inside of a fixistic paradigm can be composed, in our opinion, by postulating that the individuals generated when an operation of speciation takes place may belong *simultaneously* to more than one species.

Since there are no exceptional states in the first clause, any two individuals of a new species have necessarily a common ancestor during the speciation period.

Operations of speciation

We assume that all processes of speciation considered up to now in Biology can be uniformly described by a collection $Opspec$ of *operations of speciation*. These operations apply to one (or two) suitable species and produce one or two species. Like the operations of generation, also those of speciation are variable. Hence we can decompose any $F \in Opspec$ into its components f_i, acting at suitable states $s_i \in \mathcal{S}$.

An operation of speciation operates through a number of generations. During this process, the original species is still present, and the new species begins to appear: this is the reason why we introduced the "exceptional states" $\mathcal{E}_S$. An operation of speciation takes place during a time interval much larger than those requested for any single operation of generation. It is therefore natural to assume that the components of an operation of speciation applied to some species X act at states that in some sense *include* all exceptinal states of the set $\mathcal{E}_X$. Of course, our states are not, in general, *pure time* intervals, and we consider a generalized inclusion.

AXIOM 7 *$Opspec$ is a collection of variable operations (the operations of speciation):*
1. if X, Y are species, $F \in Opspec$, and $FX = Z$ (or $FXY = Z$), then Z is a set of species, which in general contains at most two species;
2. if z is an individual of a species $S \in Z$, then z has an ancestor of the species X (and also one of Y, when $FXY = Z$).
3. Let $f_1, \ldots, f_n$ be the components of $F \in Opspec$, and let X, Y be species such that $FX = Y$. Let $g_1, \ldots, g_m$ be the components of $G \in Opgen$, and let x, y be individuals of the species X such that $Gx = Z$ (or $Gxy = Z$). Assume that x is involved by some f_i at the state t_i, and that in the process of applying G to x (or to x, y) it appears a component $g_j V_{s_j}^j = V_{s_j}^{j+1}$: then $Rincl\ t_i, s_j$ and $s_j \in \mathcal{E}_X$. Moreover any $z \in Z$ may belong also to the species Y.

In general the operations of speciation are simple. Binary operations occur in the case of speciation by hybridization, which is extremely rare in animals but somewhat more frequent in plants. We take the result of an operation of speciation to be a *set* of species, rather than a single species, so as to comprehend all cases occurring in practice: after the operation a new species can replace the old one, or it can add itself to the old one, or else the old species disappears and two new species take place instead.

Morphological qualities

The most effective techniques for characterizing species (without using cross reproduction) are based on *morphological criteria*. Any species is identified by isolating some morphological characters which identify and distinguish it. We introduce the notion of *morphological quality*, intended in a rather wide sense, as some physical characteristic feature, which can be observed and described, even if not necessarily connected with "how the object looks like".

We fix a large collection $Morph$, intended to contain all morphological qualities which are considered at present together with all such qualities which can possibly be considered in future developments. Clearly the whole collection $Morph$ is never conceived as completely defined, since it is continuously enriched, according to the advances of experimental Biology. This openendedness can help in finding ever new characteristic qualities. In fact, we are ready to accept morphological qualities corresponding to any visible (or better *describable*) aspect, such as the number and shape of chromosomes, or the presence/absence of fixed sequences of DNA (this can be helpful in meeting the case of the so called "sibling species"). For every living object we can find a number of qualities from the collection $Morph$, so as to completely identify it. Some of these qualities are shared by all the objects of the same species.

AXIOM 8 *$Morph$ is a collection of qualities. For any species S there exists a finite family $Morph_S$ of subsets of $Morph$ such that an individual belongs to the species S if and only if it has all the qualities of some set $M \in Morph_S$.*

This axiom is unproblematic in many cases, and in fact very often *just one* set of morphological qualities suffices, and $Morph_S$ can be taken to be the single set $M_S = \bigcup Morph_S$. The set M_S, containing all qualities which are needed for identifying all elements of S, is called the set of the *characteristic qualities* of the species S. However associating to each species a single set of morphological qualities may be unsuitable in some cases, e.g. in presence of strong sexual polymorphism, and it may happen that no individual shares all the characteristics of the species. Keeping together a family of sets, which in most cases share many common qualities, allows for taking into account such morphological differences inside a single species.

THE SCHIAPARELLI MODEL

Schiaparelli proposed a geometric model of the morphology of all animals and plants, and suggested a "fundamental formula" describing and controlling the evolution of any form of life. Schiaparelli assumes that the collection of all living objects is represented by a finite (albeit very large) number of points of a space $\mathbf{R}^n$. The dimension n corresponds to the number of biological characteristics which are sufficient to determinate uniquely each individual living object (past, present and future), the coordinates of the corresponding point being given by the values of these characteristics at any instant. (It is subsumed that these characters are passible of *measurement with real values*, a hypothesis not far from our idea that morphological qualities are physically describable.) This model fits very well with the morphological characterization of species, therefore we engraft onto our axiomatic framework the main ideas laying at the basis of the model, so as to obtain a more accurate description of the organization and evolution of the morphology of living objects.

Proposition 1 (Law of discreteness) *The complex of all living objects is (at any moment) a finite, hence discrete set of points $L \subset \mathbf{R}^n$. In particular there is a minimal distance between different points of L.*

Being an astronome, Schiaparelli conceived a similarity between the distribution of the stars in the sky and that of the points of his model. Like stars are organized in relatively dense galaxies, separated by a large amount of almost empty space, thus living objects are contained in mutually disjoint regions, and they are dense in the middle and very rare near the boundary of each region. Thus the species are naturally identifiable exactly as *galaxies* are:

Proposition 2 (Law of distinction of species) *There exists a cell decomposition of the space $\mathbf{R}^n$ such that in each n-dimentional cell the points of L located near the boundary are relatively few, and become fewer and fewer as the distance from the boundary decreases.*

It seems natural to extend the astronomic analogy to larger aggregates: thus one could imagine complexes of cells corresponding to *classes, phyla, etc.*, and state the coresponding propositions. But Schiaparelli goes also the other way on, and he isolates one point at the center of each cell, to be taken as the "normal type" of that cell (note that this point itself may not belong to L). He claims that a great distance from the normal type is very unlikely in any process of generation, and in most cases this distance has an *external* reason). It follows that a statistical law distributes the points of L in decreasing density, depending on the distance from the normal type.

Proposition 3 (Law of the distribution around normal types) *The center of any cell is called its normal type. A point rather far from the normal type can be produced inside the cell only under the influence of unlikely combinations. In any cell the density of L decreases with the distance from the normal type.*

In order to give a better justification and a possible quantification of the statistical distribution stated above, Schiaparelli introduces an *attractive force* which is inversely proportional to the distance from the normal type:

Proposition 4 (Law of attraction of normal types) *Points far from the normal type tend to produce new points closer to it: in fact, there is an attractive force increasing as much as the distance from the normal type increases.*

Proposition 4 provides a great stability of any species, and in fact even the possibility of *speciation* seems excluded. But now Schiaparelli introduces his *fundamental law*, allowing for the persistency of *oriented perturbations*. Each generation has corresponding vectors connecting the parents with the offspring: although the length of these vectors are usually small (according to the precending laws), nevertheless, if a persistent perturbation contrasts the attraction of the normal type and maintains the vectors pointing in the outside direction, then after several generations part of a population can leave its original cell.

Proposition 5 (Fundamental law of speciation) *If some persistent (natural or atificial) perturbation maintains its direction through many generations, then the differences from the normal type can add one to another, so as to reach eventually a previously not inhabited region.*

We conclude this short review by an important remark: the evolution considered by Schiaparelli is different from that of Darwin in a crucial point. The transformations of living objects are not completely casual in the large: the small, casual modifications have to be summed up in such a way that the cell (type) reached at the end of the process is not really new and indeterminate, but is chosen among a *restricted range of possible types* (and it is, in principle, the *most convenient one*).

An axiomatic engrafting

We suppose that the collection $Morph$ is finite, and we assume that to each morphological quality $q \in Morph$ it is associated a fixed physical characteristic, passible of evaluation in a suitable metric space R_q (which we construe as subspace of an euclidean space $\mathbf{R}^{n_q}$, for sake of simplicity).To any "concrete" object x is associated a q-value $\eta_q(x) \in M_q$ (the *measure* of x w.r.t. the quality q). Similarly, to any variable object X is associated a q-*trajectory* $\eta_q^X = \{(s, \eta_q(X_s) \,|\, s \in \mathcal{S}\}$, which describes the variation of the q-values of (the instantiations of) X, depending on the states of the universe. The idea is that a living object X *has* the morphological quality q if and only if, for a suitable set of states $\mathcal{N}(X)$, the set of the q-values of its instantiations, i.e.

$$N_q^X = \eta_q^X(\mathcal{N}(X)) = \{\eta_q(X_s) \,|\, s \in \mathcal{N}(X)\}$$

is included in a selected compact subspace $K_q \subset R_q$ (K_q is the *distingueshed* subspace of *acceptable* values, $\mathcal{N}(X)$ is the set of *normal* states for X, and N_q^X is the *normal* q-*trajectory* of X).

The morphological characterization of species can be made measurable by combining Schiaparelli's *normal type* with our set of *characteristic qualities*. In this way we reduce the enormous number of dimensions of the entire product space

$$\mathcal{R} = \prod_{q \in Morph} R_q,$$

which is the correspective in this setting of the euclidean space $\mathbf{R}^n$ of Schiaparelli's model. The region of $\mathcal{R}$ corresponding to any species S is determined by its projection onto the more tractable reduced product

$$\mathcal{R}_S = \prod_{q \in M_S} R_q.$$

We consider the simplest case $Morph_S = \{M_S\}$: the general case is quite similar (simply take a polymorphic species as a *cluster* of monomorphic ones). More precisely, we fix a *normal type* for each species and postulate that any member of that species, at its normal states, takes on values *sufficiently close* to the normal type. Moreover, the density of the points representing members of the species in their normal states decreases when the distance from the normal type increases, and tends to zero in a neghborhood of the boundary.

AXIOM 9 *For any $q \in Morph$ there is an operation η_q taking on values in a metric space R_q (η_q is the q-measurement or q-evaluation).*
1. For any living object X there is a set of states $\mathcal{N}(X) \subset \mathcal{S}$ such that $N_q^X = \{\eta_q(X_s) \,|\, s \in \mathcal{N}(X)\}$ is a compact subspace of R_q for all $q \in Morph$ ($\mathcal{N}(X)$ is the set of normal states for X and N_q^X is the normal q-trajectory of X).
2. For any $q \in Morph$ there is a compact subspace $K_q \subset R_q$ such that a variable object X has the quality q if and only if $N_q^X \subseteq K_q$ (K_q is the distingueshed subspace of R_q).

AXIOM 10 *For any species S there exist a subset $M_S \subset Morph$, whose elements are the characteristic qualities of S, and a point $\pi_S \in \mathcal{R}_S = \prod_{q \in M_S} R_q$ (the normal type of S), such that:*
1. there is a real number r_S such that a living object X belongs to the species S if and only if $\prod_{q \in M_S} N_q^X \subseteq B(\pi_S, r_S)$ (r_S is the normal radius for S);
2. for any $s \in \mathcal{S}$, and any $d < r_S$ put $L_S(s,d) = \{\, X \,|\, s \in \mathcal{N}(X),\, (\eta_q(X_s))_{q \in M_S} \in B(\pi_S, r_S) \backslash B(\pi_S, d)\}$: then the function $\lambda_S(s,d) = |L_S(s,d)|$ tends decreasingly to 0 as d tends to r_S, uniformly in $s \in \mathcal{S}$.

AXIOM 11 *Let X belong to the species S, and let G be an operation of generation such that $GX = \{Z^1, ..., Z^n\}$. Put*

$$\xi = \text{mean value } \{ (\eta_q(X_s))_{q \in M_S} \,|\, s \in \mathcal{N}(X) \}$$

and, for $i = 1, \ldots, n$

$$\zeta^i = \text{mean value } \{ (\eta_q(Z^i_s))_{q \in M_S} \,|\, s \in \mathcal{N}(Z^i) \}.$$

Then the legth of the vector $\xi\zeta^i$ is bounded by a constant $k \ll r_S$, and its direction depends on a force pointing to the normal value π_S, whose intensity is inversely proportional to the distance betweemξ and π_S.
Moreover, if F is an operation of speciation such that $FS = S'$, and G is performed during the interval of speciation (i.e. at states included in those of $\mathcal{E}_S$), then there is also a force pointing in the direction of π'_S, whose intensity is directly proportional to the distance betweemξ and π_S.

One could have been more precise about the size of the function $\lambda_S(s, d)$, but any reasonable estimate should depend on a *quantitative* formulation of the Axiom 10, which is far from being available up to now.

In fact, our axioms give only a *geometric* (or better *cinematic*) representation of the evolution of the morphological characters. We can obtain a *dynamic* treatment of evolution by introducing two contrasting forces: an *attractive* one, which is always active and tends to maintain the trajectories close to the normal type, and another force, which is sensible only during the speciation periods, and tends to direct the trajectories towards a new, previously empty confining region. These forces act at each generation, each one trying to move the offspring in the preferred direction.

For sake of simplicity, we formulated the axiom for a *simple* operation of generation; however it becomes even more interesting in case of *binary* (sexual) generations, since then one has to take into account the reciprocal positions of both parents in the space $\mathcal{R}_S$ (and possibly also the states). Similarly, if the operation of speciation considered in the axiom preserves the original species, or produces two new species, then the perturbing force is dependent on the states.

The axioms above are only a first, limited step towards the elaboration of a general theory of the evolution of species which should combine the flexibility of the *qualitative* approach *à la* De Giorgi with the analytical-geometrical approach *à la* Volterra-Schiaparelli. They should allow for limited previsions (or "postvisions"), so as to permit a better evaluation of the correspondence between the axiomatic framework and the experimental and historical data.

At the moment these axioms pose much more problems than they can solve, e.g.:

- which collections are $Morph$ and M_S, and which spaces R_q have to be choosen?
- how can one identify the normal types and the normal radii?
- what kind of forces are those claimed for in the Axioms above?

References

[2000]E.De Giorgi, M.Forti, G.Lenzi - Verso i sistemi assiomatici del 2000 in Matematica, Logica e Informatica, *Nuova civiltà delle Macchine* **57-60** (1997), 248-260.

[GF97] L. Galleni, M. Forti - Un'assiomatizzazione di alcuni concetti della Biologia nell'ambito delle teorie fondazionali di E. De Giorgi, in *La matematizzazione della Biologia: storia e problematiche attuali - Atti del I workshop di Arcidosso, 31/VIII-3/IX 1997* (P. Cerrai and P. Freguglia, eds.). QuattroVenti, Urbino 1999, 119–132.

[Fo00] M. Forti - The Foundational Theories of Ennio De Giorgi, in *Foundations in Mathematics and Biology: Problems, Prospects, Interactions* (G. Basti, L. Galleni, A. Perrone, eds.), *Aquinas* **43** (2000), 355-367.

[GF99] L.Galleni, M.Forti - An axiomatization of biological concepts within the foundational theory of Ennio De Giorgi, *Biology Forum* **92** (1999), 77-104.

[FFG]M. Forti, P. Freguglia, L. Galleni - An axiomatic approach to some biological themes, in *The applications of Mathematics to the sciences of nature: critical moments and aspects - Proceedings of the I international workshop, Arcidosso, 28/VI-1/VII 1999* (P. Cerrai, P. Freguglia and C. Pellegrini, eds.). Kluwer Academic/Plenum Publshers, New York 2002, 139–154.

[GF00] M. Forti, L. Galleni - *Un'assiomatizzazione della genetica all'interno della teoria dei fondamenti di Ennio De Giorgi*. Quad. C.I.S.S.C. **3**, Pisa 2000.

[VGF] C. Veracini, L. Galleni, M. Forti - The concept of species and the foundations of biology, a case study: the *Callithrix jacchus* group (Primates-Platyrrhini). *Biology Forum* **95**(2002), 75–100.