The Foundational Theories of Ennio De Giorgi[1]

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ABSTRACT

We propose a non-reductionist, self-descriptive, open-ended axiomatic framework for the Foundations of Science developed by the author within the foundational programme of Ennio De Giorgi. This framework is based on the primitive "premathematical" notions of quality and relation, as in [4-10]. We introduce also the fundamental notions of operation, collection and set, proposition and predicate. A very strong selfdescriptive power is obtained by means of the quality of being a true proposition together with an operation generating predicates and propositions which describe the behaviour of any kind of objects.

1. Introduction

In the late seventies, Ennio De Giorgi started a general research programme on the Foundations of Mathematics, which became the subject matter of a weekly Seminar at the Scuola Normale Superiore in Pisa, directed by him till his death in october 1996. During the years the scope of the Seminar widened to encompass Logic, Computer Science and other sciences, according to the interests of students and scholars attending it. In this seminar, Foundations were not intended to establish *safe and unquestionable grounds* to *scientific activity*, but rather to provide *conceptual environments* where this activity could be carried out naturally and without artificial constraints (see [1],[2],[5]).

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De Giorgi used to stress the *sapiential* aspects of this kind of foundational research, and conceived it not for *specialists* in Foundations only, but rather for all interested scholars (see [4]). To this aim, an exposition in a rigorous yet informal style seemed more appropriate than a complete formalization. The traditional *contentual* axiomatic method of classical Mathematics facilitates the analysis of different notions and makes it possible to formulate proposals and submit them to criticism and contributions of interested scholars from different fields.

This is a significant point of difference between the existing foundational frameworks, which come as *formal systems,* and theories *à la* De Giorgi, which are intended as *axiomatizations* of many different concepts, which *can*, but not necessarily *have to be*, specified in any formal system. We do not intend to propose *new* formal systems, which should have, because of some technical feature, a stronger expressive power than existing logical frameworks such as First or Higher Order Logic, λ-calculus, Type Theory, Set Theory, Category Theory. The expressive power of our theories derives rather from the possibility of *accommodating naturally* all these framework theories, without having to utilize arbitrary encodings. Hence it seems inappropriate to compare f*ormally* our foundational theories to existing ones: our goal is that of *engrafting,* and hence of encompassing, rival theories, not that of *superseding* them.

Another important feature of De Giorgi's foundational theories is the absence of *constructivist* tenets. Although we believe that constructive mathematics is one of the highest achievements of this century, still we think that even constructive mathematics itself can benefit by serious research stemming from different philosophical standpoints. Banning certain methods would merely impose useless constraints to scientific activity.

We can summarize in four points the principles which inspire and inform De Giorgi's foundational programme:

- *Non-reductionism*: there are many kinds of "qualitatively different" objects and concepts which are studied by science. All

codings are ultimately artificial and opaque, and they can undermine the conceptual clarity of the notions under consideration. Even in Mathematics, where the set-theoretic encoding has been fruitful in providing a deep logical analysis of many concepts, nevertheless blurring the distinction between *natural mathematical concepts* and their usual *set-theoretic implementations*, makes it difficult, even impossible at times, to formulate appropriate axioms and conjectures. E.g. the intuitive notion of *operation* brings about the *non-extensional* concept of computation process, and so operations cannot be accounted for only in terms of graphs. Similarly, conceiving *collections* as t*ruth-valued operations* forces unnecessary commitments on the definition of collection, and yet it does not make apparent the crucial property of collections which is *extensionality*. Taking *natural numbers* as primitives saves us from having to fix priorities among different implementations, such as Frege-Russell cardinals, Von Neumann ordinals, Church numerals, etc.. Last but not least, the possibility of using different kinds of objects makes it possible to indroduce many otherwise "dangerous" selfreferential objects, and simplifies the enterprise of engrafting new notions from different disciplines.

• *Open-endedness*: a foundational theory is a framework suitable for accommodating in a natural way most of classical and modern theories arising in Mathematics, Logic, Computer Science, and possibly other sciences (Biology, Economics, Linguistics, etc.). Any sufficiently clear concept should be directly "engraftable"(*innestabile*) in it, by introducing suitable kinds of objects together with relations and operations involving them. For example, our axiomatic frameworks have been used for a detailed treatment of many mathematical concepts (*collections, sets* and *functions, natural* and *real numbers, operations, categories, etc*., see [6], [8]). Also the concept of *variable* of classical Mathematical Physics and of Economics, as well as meta-mathematical and logical notions such as *judgements, contingent qualities* and *modalities* have already been engrafted ([3],[4],[7]). Moreover some basic notions from Biology (*living objects, species,*

operations of generation and *speciation*) have been recently considered within these frameworks, see [9,11].

• *Selfdescription*: the most relevant properties, operations and relations which are utilized by a foundational theory should themselves be "first class" objects of the theory. In particular one should introduce various *qualities* classifying the different *kinds* of objects, together with *relations* and *operations* describing their behaviour. Also the *assertions* which express the main facts of the theory have to be objects of the theory.

• *Semiformal axiomatization*: a foundational theory should be expounded in a rigorous yet informal style, and investigated using the axiomatic method of traditional mathematics. Moreover, in order to keep the metatheoretic requirements to a minimum, a foundational theory should be finitely axiomatized. Although many formalizations can be given within usual systems, none can be satisfactorily taken as definitive. In fact, any formalization misses part of the intended meaning of the original theory. However, several formalizations have been carried out in First Order Logic by different authors, and they provide very powerful and comprehensive formal systems.

There are various different ways of choosing the basic primitive notions of such an axiomatic framework. Since most disciplines in Science and Humanity, deal with *several qualitatively different* objects and study *relations* over them, it seems appropriate to propose, as starting point, a general axiomatic framework based on the primitive "premathematical" notions of *quality* and *relation*. We introduce in Section 2 a few basic qualities and relations, and we use them for axiomatizing other fundamental notions occurring in all scientific theories, namely those of *operation, collection* and *set*. Additional qualities, relations and operations peculiar to each specific field can then be naturally engrafted within this framework, following [6]. We provide in Section 3 *propositions* and *predicates* suitable for describing the behaviour of *all* qualified kinds of objects, both already introduced and to be engrafted later on. We introduce also a few logical operations, and, most important, the quality of being a *true*

proposition. In Section 4 we face the problem of the twofold character of the objects considered by applied sciences: a"permanent abstract" thing versus a "concrete body at a given instant". We give only a first hint by introducing the notion of *instantiation of a variable object at a state of the universe*. Especially in this topic, sketched in [11], we hope we can foster reaction and contribution from interested researchers.

The foundational theory thus obtained can appear *prima facie* unnecessarily complicated; but this fact seems unescapable, if we want to represent *naturally* within this framework, at least in principle, the multitude of conceptual systems. De Giorgi used to quote Hamlet in order to epitomize this inexhaustible complexity:

> There are more things in heaven and earth ... than are dreamt of in your philosophy. (Act I, Sc.V, vv.166-167).

2. The basic framework

We start from the observation that Mathematics, Logic and Computer Science, as well as most disciplines in Science and Humanity, deal with several *qualitatively different* objects and study *relations* over them. It seems therefore appropriate to propose a general axiomatic framework based on the primitive notions of "having a given quality" and "being in a given relation with" and to introduce first a few fundamental qualities and relations.

Following [5] and [6], we isolate as *primitive*, i.e. not reducible to preceding concepts, the following notions:

- the object q is a *quality,*
- the object x *has the quality* q (written as qx);
- the object r is a *binary relation*,
- the objects x,y a*re in the binary relation* r (written as $r\,x,y$);
- the object s is a *ternary relation*,
- the objects x,y,z *are in the ternary relation* s (written as $s\,x,y,z$);

- the object *t* is a *quaternary relation*,
- the objects *x,y,z,w are in the quaternary relation t* (written as *t x,y,z,w*).

We do not exclude, of course, that there are also *quinary* relations, and in general relations of any larger arity. The reason for stopping here to quaternary relations is twofold. On the one hand, in this paper we utilize only relations with arity up to four. On the other hand, we could have considered only binary relations and used some internal encoding of *n*-tuples, as it is usual in formalized Mathematics. However, it seems more in line with our informing principles not to have to encode natural primitive concepts such as *simple* and *binary operations*, which are *naturally* described below by means of a ternary and a quaternary relation. Notice that, in order not to have an unnecessary constraint, we adopt the traditional philosophical attitude in dealing with qualities and relations as *intensional notions,* and we *do not* postulate *extensionality* for them.

We introduce first four distinguished qualities corresponding to the primitive kinds of objects considered above, i.e. the *classifying* qualities: *Qqual, Qrelb, Qrelt, Qrelq*.[2] Hence we postulate:

AXIOM Q. *Qqual, Qrelb, Qrelt, Qrelq are qualities.*

1) No object has simultaneously two of the above qualities;

2) x is a quality if and only if Qqual x;

3) x is a binary relation if and only if Qrelb x;

4) x is a ternary relation if and only if Qrelt x;

5) x is a quaternary relation if and only if Qrelq x.

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[2] As an aid to recalling the intended meaning of the distinguished objects we use *suggestive names*, containing fragments of the corresponding *Latin* word. E.g. qualities have prefix *Q*, relations *R*, qualities and relations concerning qualities or relations usually have suffix or infix *qual* and *rel*, respectively. Moreover basic operations or collections have upper case initials and objects involving them have prefix, suffix or infix *op* or *coll*.

We introduce next the fundamental relations *Rqual, Rrelb, Rrelt,* which *describe the behaviour of qualities and relations*. Hence we postulate:

AXIOM R1. *Rqual is a binary relation:*

1) if Rqual x,y then Qqual x;

2) if Qqual q then Rqual q,x if and only if q x.

AXIOM R2. *Rrelb is a ternary relation:*

1) if Rrel b x,y,z then Qrelb x;

2) if Qrelb r then Rrelb r,x,y if and only if r x,y.

AXIOM R3. *Rrelt is a quaternary relation:*

1) if Rrelt x,y,z,w then Qrelt x;

2) if Qrelt s then Rrelt s,x,y,z if and only if s x,y,z.

Apparently the action of quaternary relations is left here without internal description: it would require a *quinary* relation *Rrelq*, thus giving rise to an infinite sequence of fundamental relations of increasing arities, as in [2]. As pointed out above, we prefer to stop at a low level of complexity, since an internal description of quaternary relations can equally well be obtained in various ways, say following [5] or [8]. Here (in Section 3), we shall choose the more natural and direct description given in [6] and [10], and we shall provide *predicates* and *propositions* dealing with any *relation* involving objects .

We conclude this section by considering two other primitive kinds of objects, which occur in developing any scientific theory: namely *operations* and *collections* (or sets). According to the basic pattern for engrafting new concepts in our framework, we introduce suitable qualities, which classify the objects under consideration, and suitable relations, which describe their behaviour and connect them with previously considered objects. Also operations and collections will be

useful in engrafting new theories, after being introduced below. However we shall maintain the guiding principle that the "most fundamental" (hence less "manageable" and more "dangerous") objects are qualities and relations.

The concept of *operation* builds solely on the intuition of an operation as an object which *acts* (operates) on one, or two, objects and possibly produces a *result*. As in the case of relations, we do not exclude that there are operations which act on more than two objects at once: we simply do not need to consider them here. Hence we introduce the qualities *Qops* and *Qopb* of being respectively a *simple* (unary) *operation* and a *binary operation* and the corresponding relations *Rops* and *Ropb*. The *functionality* of operations is expressed by postulating that the relations *Rops* and *Ropb* are "univalent". Hence we postulate:

AXIOM O1. *Qops is a quality, Rops is a ternary relation:*

1) if Rops x,y,z then Qops x;

2) if Rops x,y,z and Rops x,y,w then z=w.

AXIOM O2. *Qopb is a quality, Ropb is a quaternary relation:*

1) if Ropb x,y,z,w then Qopb x;

2) if Ropb x,y,z,w and Ropb x,y,z,t then w=t.

The concept of *collection* aims to capture the most general notion of "aggregation into a whole of different objects of any kind", which underlies the definitions of *class* and *set*, given by Frege and Cantor. They are "extensional" in nature, in the sense that collections having the same members are identical. On the contrary, we have *not* postulated extensionality for operations, since we do not want to rule out the possibility that there exist operations, acting on the same objects and giving the same result on each object, which are nonetheless different, for they operate according to different procedures. Hence we introduce the quality *Qcoll*, of being a collection, together with the binary relations *Rcoll* (*membership*) and *Rincl* (*inclusion*), and we postulate:

AXIOM C1. *Qcoll is a quality; Rcoll and Rincl are binary relations:*

1) if Rcoll x,y then Qcoll x, i.e. x is a collection;

2) if C,D are collections then Rincl C,D holds if and only if Rcoll D,x implies Rcoll C,x for any x;

3) if C,D are collections and both Rincl C,D and Rincl D,C hold, then C=D.

Given the axioms above, we can adopt unambiguously the standard notations:

• if *f* is a simple operation, then *y=fx* stands for *Rops f,x,y;* if there exists an object *y* such that *y=fx* we say that "*f is* defined at *x*", or that "*fx* exists", and that "*y* is the result of *f* applied to *x*"; moreover we denote such a *y* by *fx*. If no such *y* exists, then we say that "*f* is undefined (or not defined) at *x*".

• if *g* is a binary operation, then *z=gxy* stands for *Ropb g,x,y,z*; if there exists *z* such that *z=gxy* we say that "*g* is defined at *x,y*", or that "*gxy* exists", and that "*z* is the result of *g* applied to *x,y*"; moreover, we denote such a *z* by *gxy*. If no such *z* exists, we say that "*g* is undefined (or not defined) at *x,y*";

• if *C* is a collection, then we write *x C* for *Rcoll C,x* and we say that "*x* belongs to (is an element of) *C*"; we write *x C* when it is not the case that *Rcoll C,x*. If *C,D* are collections, then we write *C D* (or *D* ∏ *C*) for *Rincl C,D,* and we say that *C* includes *D,* or that *D* is a subcollection of *C*.

Here we do not postulate the existence of any fundamental operation. We postulate only the existence of few collections, namely the *universal* collection *V*, the *empty* collection Ø, the collection of *all* collections *Coll,* and the collection of all sets *Ins*[3].

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[3] The word *Ins* has been chosen in accordance with the Latin *insimul* (meaning *simultaneously*) which is the root of the word for *sets* in some Neo-Latin languages, e.g. *insieme* (Italian), *ensemble* (French)

AXIOM C2. *V, Ø, Coll, Ins are collections:*

1) for all objects x, x V and x Ø;

2) x Coll if and only if Qcoll x;

3) Ins is a subcollection of Coll, whose elements are the sets.

Sets are selected as "manageable small collections", which correspond to the usual collections considered in applied science, and can be dealt with freely by all means of ordinary Mathematics. We assume of course that all (intuitively) finite collections are sets, and that collections are closed under *relative complement* (hence also under *finite unions* and *intersections*). To this aim we introduce the operations *Diff* and *Sing*, subject to the axiom

AXIOM C3. *Diff is a binary operation and Sing is a simple operation:*

1) for all X,Y Coll Diff XY = X\Y exists and is a collection such that t X\Y iff t X and t Y;

2) for any object x, Sing x = {x} exists and is a set such that t {x} iff t=x;

3) the union of two sets and the difference between a set and a collection are sets.

Other operations, collections and sets peculiar to different sciences can be introduced in dew time. A thorough development of the theory of operations and collections is carried out in [8].

3. Propositions and predicates

In this section we deal with the logical notions of *proposition* and *predicate*, which will be used for obtaining an internal description of the whole theory. Although we are interested in particular to "factual

assertions", we consider first the "most general" notion of proposition, including commands, questions and pure nonsense, too. We characterize this notion by the quality *Qprop*; thus *Qprop x* is intended to mean that *x* is a proposition in this general sense, without further specifications.

In order to obtain factual (*predicative*) propositions, we use the traditional Fregean notion of predicate as *proposition-valued operation*. Hence we introduce the quality *Qpred* of being a *predicate*, intended as a simple total operation which, possibly after several iterations, produces a proposition asserting (or better *guessing*) something about the objects involved. For technical reasons, it seems convenient to identify the "predicative propositions" with self-reproducing predicates. We are thus lead to postulate

AXIOM P1. *Qpred and Qprop are the qualities of being a predicate and a proposition, respectively.*

1) If Qpred p, then p is a simple total operation such that px is a predicate for every x.

2) If Qpred p, then Qprop p holds iff px = p for all x.

We refer to [6] and [10] for the development of Predicate Calculus in this setting. Here we introduce only the logical operations of *conjunction* (*Et)* and *negation* (*Non*), with the natural commutation rules:

AXIOM P2. *Et is a binary operation and Non is a simple operation. Let p,q be predicates; then:*

1) Non p exists and is a predicate such that (Non p)x = Non (px) for all x;

2) Et pq exists and is a predicate such that (Et pq)x=Et (px)(qx) for all x.

We adopt the usual notations $p\&q$, $\neg p$ instead of *Et pq*, *Non p* respectively.

A characteristic feature of this theory, as developed in [6] and [10], is the presence of a quality *Qver* corresponding to *internal absolute truth*, differently from the usual practice, where truth is *external,* and also from [4] and [5], where truth is *relative* to various classes of sentences. Here *Qver p* is intended to mean that the proposition *p* is (absolutely) true with respect to the real state of affairs in the whole universe *V*. Since the quality *Qprop* can comprehend all kinds of propositions (including imperative, interrogative, dubitative propositions, etc.) it would be inappropriate to postulate the "principle of excluded middle" (*tertium non datur)* for general propositions. In the axiom below, we give only the"principle of non contradiction" and a natural "conjunction rule"for propositions of any kind.

AXIOM P3. *Qver is the quality of being a true proposition:*

1) if Qprop p, then at most one of p, $\neg$ *p has the quality Qver;*

2) if p,q are propositions then Qver (p&q) holds if and only if both Qver p and Qver q hold;

Now we can enhance the self-descriptive power of the theory, by providing uniformly predicates which describe the action of all objects, including predicates and propositions themselves. We obtain this by introducing the operation *Gap,* generating atomic predicates associated to each object. A characteristic feature of this theory, namely the fact that quaternary relations suffice for describing the behaviour of all objects, allows for an axiom that takes care of all kinds of objects, even those which have not yet been "classified".

AXIOM P4. *Gap is a simple operation. If t is a quaternary relation, then:*

1) Gap t is a predicate;

2) (((Gap t)x)y)z)w is a proposition for any x,y,z,w;

3) if t x,y,z,w holds, then Qver (((Gap t)x)y)z)w;

4) if t x,y,z,w does not hold, then Qver ¬ (((Gap t)x)y)z)w.

The clauses 3-4) postulate the *excluded middle* for all "atomic predicative propositions". In particular *(((Gap t)x)y)z)w* is true if and only if *x,y,z,w* are in the relation *t,* and the axiom above justifies the "quoted notation"

"*t x,y,z,w*"= ((((*Gap t)x)y)z)w*.

Since quaternary relations describe the behaviour of any other kind of "already qualified" objects, we can extend the quoted notation in the natural way, according to "arity" and "qualifying relations", namely:

- if *s* is a ternary relation, then "*s x,y,z*"= *((((Gap Rrelt)s)x)y)z*
- if *r* is a binary relation, then "*r x,y*"= *((((Gap Rrelt)Rrelb)r)x)y*
- if *q* is a quality, then "*qx*"= *((((Gap Rrelt)Rrelb)Rqual)q)x*
- if *C* is a collection, then "*x C*"= *((((Gap Rrelt)Rrelb)Rcoll)C)x*

Similarly we are able to extend the action of *Gap* over each kind of qualified objects, by postulating

AXIOM P5.

1) If s is a ternary relation, then Gap s = (Gap Rrelt)s

2) If r is a binary relation, then Gap r = (Gap Rrelb)r.

3) If q is a quality, then Gap q = (Gap Rqual)q.

4) If f is a simple operation, then Gap f = (Gap Rops)f.

5) If g is a binary operation, then Gap g = (Gap Ropb)g.

6) If C is a collection, then Gap C = (Gap Rcoll)C.

Putting together the axioms of Sections 2-3 one can observe an interesting phaenomenon of mutual reference. On the one hand, the quaternary relations *Rrelt* and *Ropb* describe directly the behaviour of ternary relations and binary operations, and also that of binary relations, qualities and simple operations, by passing through *Rrelb, Rqual,* and *Rops*. On the other hand, the simple operation *Gap* generates predicates producing *atomic propositions* which in turn, by means of the quality *Qver*, describe the behaviour of objects of *any kind*, including quaternary relations themselves.

This circularity makes it irrelevant the choice of stopping to quaternary relations, since an *internal* description of the action all objects is provided anyway. As it can be expected, the presence of an inner notion of truth is dangerously exposed to the known paradox of the Liar. In fact it can be seen that a complete theory of quantification would be incompatible with the classical rules of truth. As it is proved in [6] and [10], either the excluded middle fails for compound predicative propositions, or there are predicates with a true existential closure, but whithout any true instance.

The problem of finding natural strong truth axioms which do not lead to contradiction when applied to *all* predicates, as well as that of isolating natural large collections of predicates where *all classical* truth rules can consistently be assumed are interesting and difficult, and they are far from receiving a definitive answer.

4. Concluding remarks

The main problem to face in dealing with experimental sciences within a foundational framework *à la* De Giorgi, like that outlined in the previous sections, is given by the fact that the frame originates from Mathematics, Logic and Computer Science, and hence it accommodates naturally only abstract, "eternally invariant" objects. A few attemps have been made of engrafting "variation" without cumbersome parametrizations. We can refer only to the introduction of the concept of *variable* of classical Mathematical Physics suggested in [3], and to the treatment of *contingency* and *modalities* outlined in [7].

But much more is needed for superseding the obstacles met by the author in his first axiomatizations of biological notions (see [9] and [11]). We cannot go deeply into the question here, and we restrict ourselves to presenting a couple of axioms, arising from the investigations of [11]. We hope that we can obtain effective contributions from readers more acquainted with this problematic issues.

We start from the *naive* idea that the objects of natural sciences (say the *Earth*) have an individual nature wich goes beyond the singular concrete things with which they are identified at each instant (a suitable *finite aggregate of particles*). Of course, depending on the objects under consideration, not only *time* instants or intervals come into play, but also *spatial* determinations and more generally any kind of relevant *data* characterizing the given situation. Therefore we introduce a *very large* and *unspecified* collection $\boldsymbol{S}$ of parameters (the *states of the universe*), conceived to encompass all phase spaces considered by the engrafted theories, At suitable values of the parameters, any *natural* object has a *concrete* "representative", and we can consider the general relation *Rinst* connecting each such object with the states at which it is *instantiated* and with its actual representatives. Summing up, we introduce the quality *Qvar* of being a variable object and we postulate

AXIOM V1. *Qvar is the quality of being a variable object,* $\boldsymbol{S}$ *is the collection of all states, and Rinst is the ternary relation of instantiation:*

1) if Rinst x,y,z then Qvar x and $y \in \boldsymbol{S}$;

2) if Rinst x,y,z and Rinst x,y,w, then $w = z$.

If X is a variable object, s is a state, and z is an object such that *Rinst X,s,z,* then z is uniquely determined, by the clause 2) above. Therefore z will be denoted by X_s and called the *instantiation of* X *at the state* s. Whenever X has an instantiation X_s at the state s we simply say that X *exists at* s, and that X_s *is* X *at* s.

We cannot develop here in detail the theory of the connections between a variable object and its instantiations (as a matter of fact, such a theory is still largely unexplored). We consider only one example, namely the case of an (*abstract*) operation acting on a variable object and producing another variable object. It seems convenient to assume that any such operation can be "decomposed" in several (*concrete*) operations which, at suitable states, act on suitable (instantiations of) objects, and produce other (instantiations of) objects,. The first object has to be the argument and the last one the result of the abstract operation. Thus we propose the following axiom

AXIOM V2. *If F is a simple operation and X,Y are variable objects such that $FX = Y$, then there exist operations f_1 ,...,f_n, variable objects $X=Z_1,...,Y=Z_n$, and states $s_1,...,s_n$ such that $f_i(Z_i.)_{si}. = (Z_{i+1})_{si}$ for $i =1,...,n-1$.*

Obviously, similar axioms can be given for binary operations, and perhaps also for relations, qualities, or collections involving variables objects. Moreover it seems necessary to specify appropriate relations among the states involved in such axioms. But this task exceedes by far the scope of this paper, and we stop here.

REFERENCES

1. M. Clavelli, E. De Giorgi, M. Forti, V. M. Tortorelli, "A self-reference oriented theory for the Foundations of Mathematics", in *Analyse Mathématique et Applications - Contributions en l'honneur de Jacques-Louis Lions*, Gauthiers-Villars, Paris 1988, 67-115.

2. E. De Giorgi, M. Forti, G. Lenzi, "Una proposta di teorie base dei Fondamenti della Matematica,", *Rend. Mat. Acc. Lincei* (9) **5**, pp. 11-22, 1994.

3. E. De Giorgi, M. Forti, G. Lenzi, “Introduzione delle variabili nel quadro delle teorie base dei Fondamenti della Matematica,”, *Rend. Mat. Acc. Lincei* (9) **5**, pp. 117-128, 1994.

4. E. De Giorgi, M. Forti, G. Lenzi, “Verità e giudizi in una nuova prospettiva assiomatica”, in *Il fare della scienza: i fondamenti e le palafitte* (F.Barone et al., eds.).*Con-tratto*, pp. 233-252, 1996.

5. E. De Giorgi, M. Forti, G. Lenzi, “Verso i sistemi assiomatici del 2000 in Matematica, Logica e Informatica”, *Nuova civiltà delle macchine* **57-60**, pp. 248-259, 1997.

6. M.Forti, G.Lenzi, “A general axiomatic framework for the Foundations of Mathematics, Logic and Computer Science”, *Rend. Accad. Naz. Sci. XL, Mem. Mat. Appl.* **21**, pp.171-207, 1997.

7. M. Forti, F. Honsell, *Positive qualities and the ontological argument,* Quad. Dip. Mat. Appl. “U.Dini” **5**, Pisa., 1996.

8. M.Forti, F.Honsell, M.Lenisa, “Operations, Collections and Sets within a General Axiomatic Framework”, in *Logic in Florence - Selected Contributed Papers, L.M.P.S.'95* (A. Cantini, E. Casari and P. Minari, eds.), Kluwer, Amsterdam 1999.

9. M.Forti, L.Galleni, “An axiomatization of biological concepts within the foundational theory of Ennio De Giorgi”, *Biology Forum* **92**, pp. 77-104, 1999.

10. M. Forti, *Lezioni sui Fondamenti della Matematica,* Quad. Dip. Mat. Appl. “U.Dini” **11**, Pisa ,1998.

11. M.Forti, P. Freguglia, L.Galleni, “An axiomatic approach to some biological themes”, in *The applications of Mathematics to the sciences of nature: critical moments and aspects - Proceedings of the I international workshop, Arcidosso, 28/VI-1/VII 1999* (P. Cerrai *et al.*, eds.), to appear.