Truth and judgments within a new axiomatic framework*

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We have been glad to accept the proposal of writing an article for the journal "Con-tratto", because we believe that the comparison of ideas among scholars of different disciplines, and in particular between mathematicians, logicians, and computer scientists on the one hand, and philosophers and theologicians on the other hand, could be of inspiration both for the former and the latter ones. In particular we think that reflecting on the main ideas of mathematics, logic and computer science, on the very meaning of these disciplines, could be an occasion of real cultural enrichment for everybody.

The first contribution that we can give to this comparison of ideas consists in expounding, in the neatest and less ambiguous form we know, namely as an axiomatic system, what we called our "dreams" in [11] and [15]. Notice that when we state axioms we do not intend to impose them to our interlocutors, but rather to submit them to their reflexions and criticism. Actually we do not pretend to elaborate alone what will come out to be the axiomatic theories of the third millennium, rather we think that a wide comparison of ideas on this topic, open to scholars of every branch of knowledge, is the best way leading to a good grounding of mathematics, logic, and computer science, as well as of all other disciplines of science and humanities.

In mentioning dreams, we remind the words of Shakespeare's Hamlet "there are more things in heaven and earth than are dreamt of in your philosopy". These words have been elected by us as the fundamental principle of our reflexions on mathematics, logic, and computer science, and they could inspire as well all scholars of scientific and human disciplines.

We do not ignore the difficulties faced by great mathematicians, like Hilbert and Brouwer, in their attempts of finding a satisfying "philosophy", where the objects commomly considered by mathematicians can be accomodated (see [3,22,23]), but we think that it does not suit to declare an object unexisting on the sole ground that we have not found a satisfying philosophy that accomodates it. Therefore we do not accept the "formalist" or "neopositivist" solicitation to "dream less", e.g. not to assign a "too realistic" meaning to the beautiful theorems on infinite dimensional spaces of Hilbert and of Banach: on the contrary, we always recall the words of Shakespeare, and so we think that the scientist who aims to understand the real things existing in heaven and earth needs to "dream more", rather than less (see [15]). We think that the greatest achievements of science are the outcome of some "wonderful dreams": complex numbers, infinitesimal calculus, Newtonian mechanics, general relativity, quantum mechanics, etc.; we also think that a relevant part of the actual

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work of experimental physicists consists in looking for visible marks of some objects "dreamt of" by theoretical physicists.

We appreciate a neat explicit solution of a nice mathematical problem, a simple and fast computing procedure, we know that they can provide a great improvement on the mere prove of an existence theorem, but we do not believe that abandoning what Hilbert named "Cantor's paradise", peopled by more or less strange sets and by infinites of any order, may help in finding explicit solutions and efficient algorithms. Rather, we would like to imagine "Cantor's paradise" even more rich and colourful than Cantor conceived it, filled up with objects of most variate kinds: sets, collections, operations, formulae, languages with their semantical interpretations, variables, categories, standard and non-standard numbers, algorithms, informations, propositions, predicates, classical logic and other logics, etc. Admittedly, this freedom in dreaming has a corresponding duty, namely that of translating dreams into axioms, conjectures, theorems formulated with the maximal clarity and with the utmost exactitude (so as to make them, after all, understandable and subject to critical analysis also by scholars not adopting a realistic perspective, but rather a purely formalistic one). Not only the dialogue on these themes among "formalists", "realists", "idealists", "dreamers" can go on in full friendship and comprehension, but it can become a valuable source of very interesting ideas for all of them. All the axioms we shall state below are suitable to be critically discussed, and possibly modified, enriched, improved by scholars close to any current of the philosophy of science. Therefore it is important that a general argument concerning mathematics, logic, computer science be formulated in much clearer a way than a specialistic one. In fact the former argument has to be understandable and subject to criticism by a great variety of interlocutors, whereas it suffices that the latter argument be understood by a restricted group of specialists. In particular, a general talk about the main basic concepts of mathematics, logic and computer science is not reserved only to a restricted number of "specialists of foundations", but it has to be accessible to all those scholars of sciences and humanities that are informed with that sentiment the ancients called philosophy, i.e. love for Wisdom.

So we shall present, in a way that seemed to us the most accessible to scholars of diverse education, some ideas, matured in these years, about the overcoming the so called "set theoretic reductionism", i.e. the trend to reduce all of mathematics to set theory, and about the widening of the horizon of mathematics, which should describe a world much more rich and "colourful" than the world of sets, even rich and interesting in itself. The most difficult problems that we have met up to now, but perhaps also the most interesting from the philosophical standpoint, probably arise when considering the delicate connections between more "general" objects, like qualities and relations, and more "specific" objects of mathematics and logic, like operations, collections, correlations, and propositions. The central knot in these problems comes out in considering the quality *Qtver*, enjoied by *all* true propositions. We found many "technical" difficulties in searching for good axioms concerning *Qtver*; we suspect that these difficulties are only the "face appearing to a mathematician" of deeper philosophical difficulties, connected to ancient and modern antinomies, to the problems of "selfreference" (see [13,23]), to questions about the nature of "mathematical, logical, and informatical entities", to the meaning of the words "exist", "existence", which perhaps are used by a mathematician in a

different way from the philosopher, the theologist, the psychologist, etc. (Think of the distance separing an existence theorem from an existential problem!). In particular, interesting philosophical meanings could be assigned to the fact that all modern antinomies seem to be "daughters" of the classical antinomy of the Liar.

In this paper we try and give a first answer to these difficulties by introducing some "intermediate" qualities between the more traditional mathematical objects and the qualities and relations that are "most difficult to handle", such as the quality *Qtver* enjoied by all true propositions. Such "intermediate" qualities will be called "metaqualities", taking into account that they are involved in questions of the so called "metamathematical" type. However, before arriving to metaqualities, we have to take up from [11] some general ideas concerning qualities and relations. These ideas provide a first axiomatic base, upon which we think possible to "engraft" in a natural way the fundamental notions of many scientific and human disciplines, in particular the topics of this paper. The choice of this basis has been suggested by various reflections on the fundamental notions of mathematics, logic, and computer science (see, e.g., [1,3,4,23,28,29,31]) and by many conversations with scholars of different disciplines (mathematics, physics, logic, computer science, biology, history, philosophy, economics, theology, etc.), which revealed the convenience of overcoming all kinds of "reductionism", trying instead to accomodate the mathematical theories, and possibly also other scientific theories, within a wider framework, suitable for a critical comparison of the fundamental ideas of the different disciplines. For instance, although the theories proposed by Cantor, Russell and Whitehead, Zermelo, Church, Curry, Gödel and Bernays, Von Neumann, and other great mathematicians of this century (see [2], [4], [28]) stay among the highest expressions of the human mind in the modern era (comparable, e.g., with Newtonian mechanics, with general relativity, with Michelangelo's Moïses, with Bach's, Mozart's, Rossini's music, with Shakespeare's theatre, with the Universal Declaration of Human Rights of December 10th, 1948, etc.), nevertheless, in order to better understand the main ideas of mathematics, logic, and computer science, it seems appropriate to put them in a more general framework, ruled by the two ideas of *quality* and *relation*. In fact these disciplines, as well as physical, chemical, biological, economic, linguistic disciplines, etc., consider *qualitatively different objects*, and study *relations among these objects*. Therefore we proposed in [11] a symple system of few fundamental *qualities* and *relations* that should constitute the solid and flexible basis upon which more specific qualities and relations of various sciences could be inserted: these qualities and relations are intended as a general premise to the axiomatic treatment of mathematics, logic, and computer science and to the comparison with other scientific and human disciplines. This premise is taken up in section 1 of this paper, which we tried to write in a plain style, so as to be accessible to scholars of variate education, but, at the same time, precise and neat enaugh to avoid equivocation and ambiguity.

1. Fundamental qualities and relations

We deal in this section with the first fundamental ideas concerning qualities and relations, intended as primitive notions. Notice that with this meaning

of primitive notion we do not intend to answer the psycological problem as to which are the ideas that come first to the mind of the child, nor to the historical problem as to which have been the ideas that mankind has considered first; we simply intend that these notions are not reduced (through suitable definitions) to other previously introduced notions.

So we introduce as primitive notions the idea of "*quality*" and the idea of "*enjoying a given quality*". We stipulate that, given an object x of any kind, and a quality q, when writing

$$qx$$

we intend that "x enjoys the quality q".

In this section we introduce seven fundamental qualities, whose meanings are:

Qqual x means that x is a *quality*;
Qrel x means that x is a *relazione*;
Qrelb x means that x is a *binary relation*;
Qrelt x means that x is a *ternary relation*;
Qrelq x means that x is a *quaternary relation*;
Qrun x means that x is a *univocal relation*;
Qrbiun x means that x is a *biunique relation*.

We introduce also four fundamental relations:

Rqual, the binary relation connecting *qualities* with *objects enjoying them*;
Rrelb, the ternary relation connecting *binary relations* with *objects that are in these relations*;
Rrelt, the quaternary relation connecting *ternary relations* with *objects that are in these relations*;
Rid, the binary relation of *identity*.

The seven qualities enjoy *Qqual* and so we can write:

Axiom 1.1 – *Qqual Qqual, Qqual Qrel, Qqual Qrelb, Qqual Qrelt, Qqual Qrelq, Qqual Qrun, Qqual Qrbiun.*

The three qualities *Qrelb*, *Qrelt*, *Qrelq* are specializations of the more general quality *Qrel*, namely:

Axiom 1.2 – *An object x enjoying any of the qualities Qrelb, Qrelt, Qrelq enjoys also the quality Qrel.*

Notice that we do not exclude at all that there exist more complex relations than binary, ternary, and quaternary relations: we simply do not introduce them here, because we make no use of them in this paper.

After the primitive idea of enjoying a given quality, the second most important primitive idea of this section is that of "*being in a given relation*". More precisely, given objects x, y of any kind and a binary relation r, we write

$$r\ x, y$$

or at times

$$r\ x; y$$

to intend that "x and y are in the relation r". Instead of saying that x and y are in the relation r, we shall also say that "x is in the relation r with y". We remark that the use of a "semicolon"may be useful when the arguments are integers: for instance, if $x = 127$ and $y = 151$, writing $r127; 151$ is clearer than writing $r127, 151$, the latter hinting to the decimal notation of numbers.

Similarly, if x, y, z are objects of any kind and ρ is a ternary relation, we write

$$\rho\ x, y, z$$

or

$$\rho\ x; y; z$$

to intend that "x, y, z are in the relation ρ".

Finally, if τ is a quaternary relation and x, y, z, t are objects of any kind, we write

$$\tau\ x, y, z, t$$

or

$$\tau\ x; y; z; t$$

to intend that "x, y, z, t are in the relation τ".

We pass to the four fundamental relations *Rqual*, *Rrelb*, *Rrelt*, *Rid*, and we state the following axioms:

Axiom 1.3 – *Rqual is a binary relation. Given objects x, y, if*

$$Rqual\ x, y$$

then x is a quality (i.e. x enjoys Qqual). Moreover, if q is a quality and x is any object, then one has Rqual q, x if and only if qx (i.e. x enjoys the quality q).

Axiom 1.4 – *Rrelb is a ternary relation. Given objects x, y, z, if one has*

$$Rrelb\ x, y, z$$

then x is a binary relation (i.e. x enjoys Qrelb). Moreover, if r is a binary relation, and x, y are objects of any kind, then one has Rrelb r, x, y if and only if $r\ x, y$ (i.e. x, y are in the relation r).

Axiom 1.5 – *Rrelt is a quaternary relation. Given objects x, y, z, t, if one has*

$$Rrelt\ x, y, z, t$$

then x is a ternary relation (i.e. x enjoys Qrelt). Moreover, if ρ is a ternary relation and x, y, z are objects of any kind, then one has Rrelt ρ, x, y, z if and only if $\rho\ x, y, z$ (i.e. x, y, z are in the relation ρ).

Axiom 1.6 – *Rid is a biunique relation. In order to have*

$$Rid\ x, y$$

it is necessary and sufficient that x and y are exactly the same object. In other words, any object is in the relation Rid with itself and only with itself.

The relation *Rid* expresses the identity of an object of *any kind* with itself, and so we often write $x = y$ instead of $Rid\ x, y$.

The relation *Rid* is the simplest instance of a ralation enjoing both qualities *Qrun* and *Qrbiun*. Concerning the quality *Qrun*, we state the following axiom, which expresses the idea of *univocity* applied to *binary, ternary* and *quaternary relations*:

Axiom 1.7 – *Qrun is a quality.*
1) if Qrun x, then Qrel x;
2) if Qrelb r; Qrun r; r x, y; r x, z , then y = z;
3) if Qrelt ρ; Qrun ρ; ρ x, y, z; ρ x, y, t , then z = t;
4) if Qrelq τ; Qrun τ τ x, y, z, t; τ x, y, z, u , then t = u.

Finally *Qrbiun* is the quality of being a *biunique binary relation*, and biuniqueness is expressed by the axiom:

Axiom 1.8 – *Qrbiun is a quality.*
1) if Qrbiun x, then Qrelb x and Qrun x;
2) if Qrbiun r, r y, x, and r z, x, then y = z.

2. Operations and natural numbers

In this section we give some very simple examples of "engrafting" mathematical notions into the "trunk" the fundamental axioms concerning qualities and relations. More precisely, we introduce five qualities:

Qop, the quality of being an *operation*;
Qops, the quality of being a *simple operation*;
Qopb, the quality of being a *binary operation*;
Qtops, the quality of being a *total simple operation*;
Qnnat, the quality of being a *natural number*;

four relations:

Rops, the ternary relation describing the *action of simple operations*;
Ropb, the quaternary relation describing the *action of binary operations*;
Rnsuc, the binary relation connecting a *natural number* to its *immediate successor*;
Rtsc, the ternary relation associating to pairs of *non-zero distinct natural numbers* a *swap operation*, acting "*à la* Gödel" on *iterated total simple operations*;

and two operations:

Id, the *identity* operation;
K, the operation *generator of constant operations.*

Of course, each of these engraftings should be considered only a first "budd", from which whole "branches" of mathematics can develop in various ways. In this section we give only a few basic descriptive axioms, so as to leave open the possibility of developing the respective theories in different directions.

We begin by dealing with the notion of operation.

Axiom 2.1 – *Qops, Qopb, Qop and Qtops are qualities. Rops is a univocal ternary relation and Ropb is a univocal quaternary relation.*
1) if Qops x or Qopb x, then Qop x;

2) if Rops x, y, z, then Qops x;
3) if Ropb x, y, z, t, then Qopb x;
4) Qtops x holds if and only if Qops x and for all y there exists z such that Rops x, y, z.

So *Qops* and *Qopb* are specializations of the general quality *Qop*, while *Qtops* is a specialization of *Qops*.

When f is a simple operation, instead of writing *Rops* f, x, y we often write $y = fx$, and we say that f *acts* on x giving the *risult* y , or that f *trasforms* x into y, or that f *maps* x onto y, or also that f *applied* to x gives the *value* y. Similarly, when φ is a binary operation, instead of writing *Ropb* φ, x, y, z we often write $z = \varphi\ x, y$. We remark that *univocity* of the relations *Rops* and *Ropb* yields directly "functionality" of simple and binary operations, i.e. the fact that, if f is a simple operation acting on an object x, the result $y = fx$ is uniquely determined, and similarly, if φ is a binary operation acting on the objects x, y, the result $z = \varphi\ x, y$ is uniquely determined.

At the moment we introduce only two, very simple fundamental operations, namely Id, the *identity* operation, and K, the *generator of constant operations*, by means of the the axiom:

Axiom 2.2 – *Id and K are total simple operations.*
1) $Id\ x = x$ for any object x.
2) For any object x, the result Kx is a total simple operation such that, for any object y, one has $(Kx)y = x$.

We introduce now the quality *Qnnat* of being a *natural number* (like 0, 1, 2, 3, 4, ...) and the relation *Rnsuc* connecting each natural number to its *immediate successor*. These objects satisfy the following axiom:

Axiom 2.3 – *Qnnat is a quality, Rnsuc is a biunique relation.*
1) Rnsuc x, y implies Qnnat x, Qnnat y.
2) There exists a unique z such that Qnnat z and for no x Rnsuc x, z.
3) If Qnnat x, then there exists y such that Rnsuc x, y.

Given a natural number x, the unique natural number y such that *Rnsuc* x, y is called the *successor* of x. The axiom 2.3 suffices for characterizing, through the relation *Rnsuc*, the natural numbers 0 (the unique z of clause 2), 1 (the successor of 0), 2 (the successor of 1), etc.

The arithmetical theory provided by the axiom 2.3 is very weak, however it provides the possibility of defining *infinitely many* natural numbers, through the relation *Rnsuc*. In case only a few natural numbers are needed, e.g. only $0, 1, 2, 3, 4, 5, 6, 7$, one can replace the clause 3 in axiom 2.3 by some particular instances; e.g. one can assume the axioms:

2.7.3.1) There exists the natural number 1 *such that Rnsuc* $0; 1$*;*
2.7.3.2) There exists the natural number 2 *such that Rnsuc* $1; 2$*;*
2.7.3.3) There exists the natural number 3 *such that Rnsuc* $2; 3$*;*
2.7.3.4) There exists the natural number 4 *such that Rnsuc* $3; 4$*;*
2.7.3.5) There exists the natural number 5 *such that Rnsuc* $4; 5$*;*
2.7.3.6) There exists the natural number 6 *such that Rnsuc* $5; 6$*;*
2.7.3.7) There exists the natural number 7 *such that Rnsuc* $6; 7$.

The development of various stronger theories of (standard and nonstandard) arithmetic might depend on the introduction of operations (beginning with the

four operations of the primary school), relations (natural ordering, divisibility, etc.), collections (the collection **N** of all natural numbers, the collection **P** of all primes, etc.), of various forms of the induction principle, etc.

In this paper we do not intend to introduce the arithmetical operations (addition, multiplication, etc.) by means of corresponding axioms. We shall only employ, as a *mere notation*, the traditional expression $y = x + 1$ instead of $Rnsuc\ x, y$. From a psycological, as well as from a conceptual standpoint, passing from a number to its successor corresponds to the elementary practice of *counting*, which is definitely simpler than the practice of *adding*.

The comparison of the notions of set and of natural number is of great importance, both from a logico-mathematical and a philosophical standpoint. The classical way of carrying out this comparison is by assuming the existence of a set **N**, to which belong all and only the "true" natural numbers (i.e. those objects that enjoy *Qnnat*), and of biunique correspondences between **N** and sutable sets of sets, for instance the "finite Von Neumann ordinals"; or even by identifying **N** with one such set. We could name "neopythagorean" those expositions of arithmetic and set theory that satisfy this kind of axioms. From a logical standpoint, one could investigate "non-pythagorean" set theories, possibly connected to ancient and modern reflexions about the notions of actual and potential infinity.

We shall come back to this theme in section 6, here we conclude by using the natural numbers in introducing some "combinatorial" operations (Gödel swaps), which play a rôle in some "operational" treatments of predicate calculus, with which we shall deal in sections 3, 4, 5. Following the general line that takes qualities and relations as "more primitive" notions than operations, we introduce the swap operations by means of the univocal ternary relation *Rtsc* .

Axiom 2.4 – *Rtsc is a univocal ternary relation.*
1) If Rtsc x, y, z, then x, y are different non-zero natural numbers, and z is a simple operation.
2) If x, y are different non-zero natural numbers, then there exists z such that Rtsc x, y, z.
3) If one has Rtsc h, k, f and $y = fx$, then y, x are total operations taking on total operations as values.

The unique z such that $Rtsc\ m, n, z$ is denotated by T_{mn} or $T_{m;n}$. The swap operated by the operations T_{mn} is described by the next axiom. It is conceived in such a way that it can yield the equations:

$$((T_{1;2}f)x)y = (fy)x;$$

$$(((T_{1;3}f)x)y)z = ((fz)y)x;$$

$$((((T_{1;4}f)x)y)z)t = (((ft)y)z)x;$$

and other analogous equations of the following kind:

$$(\ldots((T_{1n}f)x_1)x_2\ldots)x_n = (\ldots(fx_n)x_2\ldots)x_1.$$

We also want the equations:

$$(((T_{2;3}f)x)y)z = ((fx)z)y;$$

$$((((T_{2;4}f)x)y)z)t = (((fx)t)z)y;$$

$$((((T_{3;4}f)x)y)z)t = (((fx)y)t)z.$$
$$(((((T_{3;5}f)x)y)z)t)s = ((((fx)y)s)t)z.$$

and other analogous equations of this kind:

$$(\dots(\dots((T_{mn}f)x_1)\dots)x_n)\dots)x_m = (\dots(\dots(fx_1)\dots x_m)\dots)x_n.$$

All these equations are consequences of the following axiom:

Axiom 2.5 –
1) $T_{mn} = T_{nm}$*;*
2) if $T_{mn}x$ *exists, then there exists also* $T_{mn}(T_{mn}x)$ *and it is equal to* x*;*
3) if m, n *are different natural numbers greater than* 1*, then* $T_{mn}x$ *exists if and only if both* $T_{1n}x$ *and* $T_{1m}x$ *exist;*
4) if both $T_{hk}x, T_{mn}x$ *exist, then there exists also* $T_{hk}(T_{mn}x)$*.*
5) $T_{1;2}f$ *exists if and only if* f *a total operation whose values are total operations, and in this case one has* $((T_{1;2}f)x)y = (fy)x$ *for all* x, y*;*
6) if f *enjoys Qtops and* m, n *are different non-zero natural numbers, then* $T_{m+1;n+1}f$ *exists if and only if* $T_{mn}f$ *esists and, for all* x*,* $T_{mn}(fx)$ *exists; in this case one has* $(T_{m+1;n+1}f)x = T_{mn}(fx)$ *for all* x*;*
7) if $T_{mh}f, T_{nh}f, T_{mn}f$ *exist, then one has* $T_{mh}(T_{mn}(T_{mh}f)) = T_{hn}f$*.*

3. Propositions and predicative operations

In this section we introduce some general ideas about *propositions* and *predicates*. More precisely, we introduce five qualities:

Qgprop, the quality of being a *proposition of general* type;
Qclp, the quality of being a *classical proposition*;
Qtgopr, the quality of being a *total predicative operation* of *general* type;
Qclopr, the quality of being a *classical predicative operation*;
Qtver, the quality (of high level) enjoyed by all *true propositions*;

a binary relation:

Rsom, the relation interconnecting *structurally homogeneous* objects;

and six operations:

Et, the binary operation of logical *conjunction*;
Vel, the binary operation of logical *disjunction*;
Non, the simple operation of *negation*;
Exist, the simple operation of *existential quantification*;
Univ, the simple operation of *universal quantification*;
Gopr, the simple operation *generator of elementary predicative operations.*

The axioms we introduce are intentionally very weak; in particular those axioms which deal with true and false propositions mostly concern "classical" propositions and predicative operations, so as to leave greatest freedom for possible engraftings of logics different from classical logic.

So we begin by introducing the "most general" notion of proposition through the quality *Qgprop*: hence we write *Qgprop* x to intend that x is a *proposition*,

without further specification. Among the propositions there are in particular "classical" propositions, subject to the rules of classical propositional calculus and characterized by the quality $Qclp$: hence $Qclp\ x$ means that x is a *classical proposition*. We can state the axiom:

Axiom 3.1 – *$Qgprop, Qclp$ are qualities. If $Qclp\ x$, then $Qgprop\ x$.*

The tradizional logical operations Et (*conjunction*), Vel (*disjunction* or *alternative*), Non (*negation*) operate on all propositions:

Axiom 3.2 – *Et and Vel are binary operations. Non is a simple operation. If p, q are propositions, then $Et\ p, q$, $Vel\ p, q$, $Non\ p$ exist and are propositions. If p, q are classical propositions, then also $Et\ p, q$, $Vel\ p, q$ are classical. Finally, $Non\ p$ is a classical proposition if and only if p is a classical proposition.*

Following the common usage, we denotere by $p\&q$, $p \vee q$, $\neg p$ the propositions $Et\ p, q$, $Vel\ p, q$, $Non\ p$.

The quality $Qtver$ denotes truth in an "absolute and total sense", therefore we write $Qtver\ x$ to intend that x is *true*, and we write both $Qgprop\ p$ and $Qtver\ p$ to mean that p is a *true proposition*. When p is a proposition and $\neg p$ enjoys the quality $Qtver$ we also say that p is *false*.

The following axiom connects classical propositions with the quality $Qtver$, expressing, *inter alia*, the classical principles of *non contradiction* and *excluded middle*:

Axiom 3.3 – *$Qtver$ is a quality. If p, q are classical propositions then:*
1) p enjoys $Qtver$ if and only if $\neg\ p$ does not enjoy $Qtver$.
2) $p\&q$ enjoys $Qtver$ if and only if both p and q enjoy $Qtver$.
3) $p \vee q$ enjoys $Qtver$ if and only if at least one of p and q enjoys $Qtver$.

Having thus introduced propositions, we can resume the classical notion of predicate "*à la* Frege" (see [19]), i.e. as an operation taking on *propositions as values*. Of course, many different introductions are possible (see, e.g., [5,12,14,24,25,26]). In this paper we consider total simple operations that produce propositions either immediately, or after some iterations. To this aim, we consider *total predicative operations* of *general type* (characterized by the quality $Qtgopr$), *classical predicative operations* (characterized by the quality $Qclopr$), and the relation of *structural homogeneity*, $Rsom$. The qualities $Qgprop$, $Qclp$, $Qtgopr$, $Qclopr$, the relation $Rsom$ and the swap operations T_{hk} ruled by the axiom 2.5 are connected to each other by the following axiom:

Axiom 3.4 – *$Qtgopr, Qclopr$ are qualities, $Rsom$ is a binary relation.*
1) For all x, y, z, if $Rsom\ x, y$ and $Rsom\ y, z$, then also $Rsom\ x, x$, $Rsom\ y, x$, $Rsom\ x, z$.
2) If α is a proposition and β is any object, then $Rsom\ \alpha, \beta$ holds if and only if β is a proposition.
3) If $Qtgopr\ f$ and g is any object, then $Rsom\ f, g$ holds if and only if one has $Qtgopr\ g$ and $Rsom\ fx, gx$ for all x.
4) $Qtgopr\ f$ holds if and only if the following condition are fulfilled:
4.1) f enjoys $Qtops$;
4.2) for all x, fx enjoys $Qtgopr$ or $Qgprop$;
4.3) for all x, y one has $Rsom\ fx, fy$.
5) If f enjoys $Qtgopr$ and $T_{hk}f$ exists, then $Rsom\ f, T_{hk}f$ holds.
6) If f enjoys $Qclopr$, then f enjoys $Qtgopr$; moreover, for all x, fx enjoys

either Qclp or Qclopr.
7) If f enjoys Qclopr and $T_{hk}f$ exists, then also $T_{hk}f$ enjoys Qclopr.

To total predicative operations apply the *existential quantifier Exist* and the *universal quantifier Univ*, which satisfy the following axiom:

Axiom 3.5 – *Univ and Exist are simple operations.*
1) If f is a total predicative operation whose values are propositions, then both $Univ\, f$ and $Exist\, f$ exist and are propositions;
2) if f enjoys Qtgopr and, for all x, fx enjoys Qtgopr, then both $Univ\, f$ and $Exist\, f$ exist, enjoy Qtgopr and one has:
2.1) $(Univ\, f)x = Univ(fx)$ for all x;
2.2) $(Exist\, f)x = Exist(fx)$ for all x.
3) If f enjoys Qclopr, then $Univ\, f$ and $Exist\, f$ enjoy either Qclopr or Qclp.
4) If f enjoys Qclopr and all values of f are propositions, then one has:
4.1) $Univ\, f$ is true if and only if the proposition fx is true for every x;
4.2) $Exist\, f$ is true if and only if the proposition fx is true for some object x.

We often adhere to the more usual notation $\forall p$, $\exists p$ instead of $Univ\, p$, $Exist\, p$; when the values of p are propositions, we use also the notation $\forall x.px$, $\forall y.py$, etc., instead of $Univ\, p$, and similarly $\exists x.px$, $\exists y.py$, etc., instead of $Exist\, p$: of course, this notation does not refer to any specific object x, y, etc.

Also the operations of conjunction, disjunction and negation act on total predicative operations, according to the following commutation rules:

Axiom 3.6 – *Let f, g be total predicative operations and assume $Rsom\, f, g$. Then*
1) $Non\, f$ exists, is a total predicative operation, and one has $(Non\, f)x = Non(fx)$ for all x.
2) $Et\, f, g$ exists, is a total predicative operation, and one has $(Et\, f, g)x = Et\, fx, gx$ for all x.
3) $Vel\, f, g$ exists, is a total predicative operation, and one has $(Vel\, f, g)x = Vel\, fx, gx$ for all x.
4) $Non\, f$ is classical if and only if f is classical;
5) if both f, g are classical, then also $Et\, f, g$, $Vel\, f, g$ are classical.

Also in the case of total predicative operations we adhere to the usual notazion $f\&g, f \vee g, \neg f$ instead of $Et\, f, g$, $Vel\, f, g$, $Non\, f$, respectively.

The links of qualities, relations, operations with the propositions describing their actions are provided by the operation *Gopr*, generator of *elementary predicative operations*. Not all operations generated by *Gopr* are classical: in fact we shall see in section 5 that *Gopr* ρ, for some relation ρ, does not enjoy *Qclopr*. The operation *Gopr* satisfies the following axioms:

Axiom 3.7 – *Gopr is a simple operation. If τ is a quaternary relation, then:*
1) $Gopr\, \tau$ is a total predicative operation;
2) for any object x, $(Gopr\, \tau)x$ is a total predicative operation;
3) for any x, y, $((Gopr\, \tau)x)y$ is a total predicative operation;
4) for any x, y, z, $(((Gopr\, \tau)x)y)z$ is a total predicative operation;
5) for any x, y, z, t, $((((Gopr\, \tau)x)y)z)t$ is a proposition;
6) for any x, y, z, t, the proposition $((((Gopr\, \tau)x)y)z)t$ enjoys Qtver if and only if one has $\tau\ x, y, z, t$ (i.e. if and only if x, y, z, t are in the relation τ).

Clause 6 of the axiom above explains the shortened notation

$$\text{“}\tau\ x, y, z, t\text{”} = ((((Gopr\,\tau)x)y)z)t.$$

Comparing the clause 6 of the axiom 3.6 with the preceding axioms, one observes an interesting phenomenon of *mutual reference*: on the one hand, quaternary relations indirectly describe, through *Rrelt*, *Rrelb*, *Rqual*, *Rops*, also qualities and simple operations; on the other hand, the simple operation *Gopr* and the quality *Qtver* describe, through the axiom 3.7, the quaternary relations. This circularity justifies the choice that we have done in section 1, of stopping at quaternary relations; these relations have also the supplementary value of describing, through *Ropb*, the binary operations.

After describing the behaviour of the operation *Gopr* at quaternary relations, it is an easy task to deal with the other kinds of objects introduced in the preceding sections. We proceed as follows:

Axiom 3.8 –
1) If ρ is a ternary relation, then $Gopr\,\rho = (Gopr\,Rrelt)\rho$.
2) If r is a binary relation, then $Gopr\,r = (Gopr\,Rrelb)r$.
3) If q is a quality, then $Gopr\,q = (Gopr\,Rqual)q$.
4) If f is a simple operation, then $Gopr\,f = (Gopr\,Rops)f$.
5) If φ is a binary operation, then $Gopr\,\varphi = (Gopr\,Ropb)\varphi$.

The “elementary” propositions generated by predicates obtained through the operation *Gopr* are statements that concern the quality or the relation to which *Gopr* has been applied. Hence we extend the quoted notation in the natural way:
if q is a quality, then “qx” stands for $(Gopr\,q)x$;
if r is a binary relation, then “$r\,x, y$” stands for $((Gopr\,r)x)y$;
if ρ is a ternary relation, then “$\rho\ x, y, z$” stands for $(((Gopr\,\rho)x)y)z)$;
finally “$x = y$” stands for $((Gopr\,Rid)x)y$,
and “$x \neq y$” stands for $\neg(((Gopr\,Rid)x)y)$.

Both the relation *Rid* and the operation Id talk about the identity of objects; hence we state the axiom:

Axiom 3.9 – *For all objects x, y one has*

$$((Gopr\,Id)x)y = ((Gopr\,Rid)x)y = ((Gopr\,Rid)y)x.$$

Now that we have introduced propositions and predicative operations, together with the logical operations Et, Vel, Non, $Exist$, $Univ$, and the “elementary predicates” generated by *Gopr*, the problem arises of searching for “axioms of classicality” that do not lead to contradiction.

Negative resultats, essentially inspired by Liar’s Paradox (or by Tarski’s theorem), have been proved in similar contexts, see [12,14], and will be discussed here in section 5. The search for the strongest positive axioms that do not lead to contradiction is still open.

4. Truth and judging subjects

We introduce in this section four qualities:

$Qjuds$, the quality of being a *judging subject*;
$Qcljq$, the quality of being a *classical judging quality*;
Qmq, the quality of being a *metaquality*;
Qmr, the quality of being a *metarelation*;

and a binary relation:

$Rjud$, the relation connecting judging subjects with objects pertaining to their judgments.

The quality $Qtver$, expressing "absolute truth", is located at a so elevated level that it is practically impoossible to subject it to the usual logico-mathematical manipulations. It seems therefore appropriate to introduce "intermediate notions" that can provide the wanted flexibility and functional capacity. So we introduce the quality $Qjuds$ of being a *judging subject*, and the corrisponding relation $Rjud$ of *bearing to judgment* by means of the axiom:

Axiom 4.1 – *$Qjuds$ is a quality, and $Rjud$ is a binary relation.*
If $Rjud\, x, y$, then $Qjuds\, x$.

Of course, the most varied kinds of judgments are possible: e.g. one may judge an object true, false, good, bad, necessary, possible, contingent, guilty, innocent, etc. In this paper we are mainly interested in judgments about truth of classical propositions, and to this aim we have introduced the quality $Qcljq$ of being a *classical judging quality*. Therefore we state the axiom:

Axiom 4.2 – *$Qcljq$ is a quality. If $Qcljq\, x$, then $Qjuds\, x$ and $Qqual\, x$.*
1) If α is a proposition and μ enjoys $Qcljq$, then:
1.1) if α enjoys μ, then α is classical and true;
1.2) $Rjud\, \mu, \alpha$ holds if and only if either α or $\neg\alpha$ enjoys μ.
2) If α, β are propositions, μ enjoys $Qcljq$, and one has $Rjud\, \mu, \alpha$, $Rjud\, \mu, \beta$, then one has also $Rjud\, \mu, \alpha\&\beta$, $Rjud\, \mu, \alpha \vee \beta$.
3) If f is a total predicative operation and μ enjoys $Qcljq$, then one has $Rjud\, \mu, f$ if and only if for all x one has $Rjud\, \mu, fx$.
4) If f is a total predicative operation, μ enjoys $Qcljq$, and one has $Rjud\, \mu, f$, then one has also $Rjud\, \mu, \exists f$, $Rjud\, \mu, \forall f$.
5) if f is a total predicative operation, $T_{hk}f$ exists, μ enjoys $Qcljq$ and one has $Rjud\, \mu, f$, then one has also $Rjud\ \mu, T_{hk}f$.

If μ enjoys $Qcljq$ and α is a proposition such that $Rjud\, \mu, \alpha$, then we say that μ *judges* α, or that α is *judged* by μ; when f is a predicative operation and $Rjud\, \mu, f$ we say that f *bears the judgment* of μ, or that μ *considers* f (in its judgments).

We now state that classical propositions and predicative operations are characterized by the fact of bearing the judgment of some classical judging quality.

Axiom 4.3 – *1) Let α be a proposition: then α enjoys $Qclp$ if and only if there exists μ, enjoying $Qcljq$, such that $Rjud\ \mu, \alpha$.*
2) A total predicative operation f enjoys $Qclopr$ if and only if there exists μ, enjoying $Qcljq$, such that $Rjud\ \mu, f$.

The connections between judging qualities, metaqualities and metarelations are established by the following axiom:

Axiom 4.4 – *Qmq and Qmr are qualities.*
1) $Qmq\, x$ holds if and only if $Qqual\, x$ and $Gopr\, x$ enjoys $Qclopr$;
2) $Qmr\, x$ holds if and only if $Qrel\, x$ and $Gopr\, x$ enjoys $Qclopr$.

In order to simultaneously judge several objects that can be "separately" judged, it is convenient to assume that, given two classical judging qualities there is a third quality which is "more competent":

Axiom 4.5 – *If μ_1, μ_2 enjoy $Qcljq$, then there exists ν enjoying $Qcljq$ and fulfilling both conditions:*
$Rjud\,\mu_1, x$ implies $Rjud\,\nu, x$;
$Rjud\,\mu_2, x$ implies $Rjud\,\nu, x$.

Finally we want to ensure that many interesting propositions and predicative operations are classical. To this aim we state:

Axiom 4.6 – *All qualities enjoying $Qcljq$ enjoy also Qmq; moreover the following qualities enjoy Qmq:*

$$Qqual, Qrel, Qrelb, Qrelt, Qrelq, Qop, Qops, Qopb, Qtops, Qgprop, Qtgopr;$$

the following relations enjoy Qmr:

$$Rid, Rops, Ropb, Rsom.$$

5. A theorem inspired by Liar's Paradox

By means of the elementary predicative operations generated by *Gopr*, and performing the manipulations allowed by the axioms of sections 3-4, one can actually express all "first order" propositions and predicates that involve objects considered in this paper. It is worth noticing that these capabilities already suffice to prove that not only *Gopr Qtver* cannot be a classical predicative operation, but also that a whole "hierarchy" of classical predicative qualities is required, a hierarchy that does not admit a dominating metaquality.

We begin by stating the following lemma:

Lemma 1. – *There exists a classical predicative operation F such that, for all x, y, $(Fx)y$ is a true proposition if and only if x is a total predicative operation, y is a proposition, and $y = \neg(xx)$.*

Proof. Put $P = Gopr\,Rops$, $Q_0 = Gopr\,Qgprop$, $Q_1 = Gopr\,Qtgopr$, and put:
$F_0 = K(K(KQ_0))$, hence $(((F_0x)y)z)t =$"$Qgprop\ t$", i.e. "t is a proposition";
$F_1 = T_{1;4}(K(K(KQ_1)))$, hence $(((F_1x)y)z)t =$"$Qtgopr\ x$", i.e. "x is a total predicative operation";
$F_2 = T_{1;2}(T_{2;3}(T_{3;4}(KP)))$, hence $(((F_2x)y)z)t =$"$Rops\ x, y, z$", i.e. "the operation x applied to y gives value z";
$F_3 = T_{2;3}(T_{1;4}(K(K(P\,Id))))$, hence $(((F_3x)y)z)t =$"$Rops\,Id, y, x$", i.e. "$x = y$";
$F_4 = K(K(P\,Non))$, hence $(((F_4x)y)z)t =$"$Rops\,Non, z, t$", i.e. "$t = \neg z$".

Now put:
$F_5 = F_0\&(F_1\&(F_2\&(F_3\&F_4)))$;
$F_6 = T_{2;4}F_5$, hence $(((F_6x)y)z)t$ says "x is a total predicative operation, y is a proposition, $t = x$, $z = xt$, $y = \neg z$";
$F_7 = \exists F_6$, hence $((F_7x)y)z = \exists t.(((F_6x)y)z)t$, i.e. "$x$ is a total predicative operation, y is a proposition, $z = xx$, $y = \neg z$";

$F = \exists F_7$, hence $(Fx)y = \exists z.((F_7x)y)z$, i.e. "$x$ is a total predicative operation, y is a proposition, $y = \neg(xx)$".

Clearly F is classical, because $Qgprop$, $Qtgopr$ are metaqualities, and $Rops$ is a metarelation. QED

In order to see how Lemma 1 and the following theorems are inspired by Liar's Paradox, we remark that, when the proposition xx is intuitively interpreted as "what x is affirming about itself", truth of both y and $(Fx)y$ yields falsity of "what x is affirming about itself".

We shall come back to this interpretation after Theorem 2; by now we show how to get from Lemma 1 a theorem analogous to Tarski's theorem (see [29]):

Theorem 1. – *Let F be as in Lemma 1. Then the predicative operation*

$$F \,\&\, K(Gopr\, Qtver)$$

cannot be classical. It follows that the quality $Qtver$ is not a metaquality.

Proof. Assume by contradiction that the operation $F\&K(Gopr\, Qtver)$ be classical; then also the operation

$$G = \exists(F\&K(Gopr\, Qtver))$$

is classical, and one has, for all x, $Gx = \exists y.((Fx)y \,\&\, \text{“}Qtver\, y\text{”})$, i.e. "$x$ is a total predicative operation and $\neg(xx)$ is a true proposition".

Taking $x = G$, one gets in particular that GG is a true proposition (it enjoys $Qtver$) if and only if G is a total predicative operation and $\neg(GG)$ is a true proposition (enjoying $Qtver$). But this is impossible if GG is a classical proposition, for in that case one and only one between GG and $\neg(GG)$ enjoys $Qtver$. QED

As one can see by looking at axiom 3.7, in order to produce the operation $Gopr\, Qtver$ one can make use of the elementary predicative operations $Gopr\, Rrelq, Gopr\, Rrelt, Gopr\, Rrelb, Gopr\, Rqual$; so Theorem 1 yields immediately

Corollary. – *The relations $Rrelq, Rrelt, Rrelb, Rqual$ are not metarelations (i.e. they do not enjoy Qmr).*

We can apply Lemma 1 also to prove that there exists no "dominating" classical judging quality, since no classical judging quality can judge *all* classical propositions.

Theorem 2. – *Let F be as in Lemma 1 and let μ be a classical judging quality considering F, i.e. such that $Rjud\,\mu, F$. Put*

$$H = \exists(F\&K(Gopr\,\mu)).$$

Then the classical proposition HH is false, but it is not judged by μ.

It follows that μ does not consider the predicate $Gopr\,\mu$, i.e. $Rjud\,\mu, Gopr\,\mu$ does not hold.

Proof. One has, for all x, $Hx = \exists y.((Fx)y \,\&\, \text{“}\mu y\text{”})$, i.e. "$x$ is a total predicative operation and $\neg(xx)$ is a proposition enjoying μ".

Then, putting $x = H$ we get:
i) if $\mu(HH)$ holds, then HH is a true classical proposition, hence $\neg(HH)$ enjoys μ, contradiction.

ii) if $\mu(\neg(HH))$ holds, then HH is a false classical proposition; so, being H a total predicative operation, the proposition "$\mu(\neg(HH))$" must be false, contradiction.

So it is impossible for μ to judge HH, i.e. $Rjud\,\mu,(HH)$ does not hold. Therefore μ does not consider the predicate H and, given that it considers F by hypotesis, it cannot consider $Gopr\,\mu$. On the other hand, *i*) implies that HH cannot enjoy $Qtver$, and so, being a classical proposition, it is false. QED

Coming back to the intuitive analogy with Liar's Paradox, we could call the operation H of Teorema 2 a "μ-liar", and interpret the proposition HH as "what H says about itself". Then we could say that H says simultaneously that it is lying and that it is judged by μ as a liar; the first statement is true, the second one is false, and so their conjunction is false, by axiom 3.3. Intuitively we could say that the "μ-liar" lies, but μ is unable to belie it!

6. Final remarks

Sections 2, 3, 4 of this paper constitute the engraftings of some mathematical and logical objects into the trunk of the fundamental qualities and relations introduced in [11] and resumed in section 1. The theorems of section 5 represent the "testing" of these engraftings by facing the crucial antinomy of the Liar.

Many more engraftings are possible, and all ideas presented up to now can be developed with great freedom, in several different directions. For instance, among the judging subjects one could consider, besides judging qualities, also *judging operations* or *relations*; one has only to introduce boolean operations or univocal boolean relations, associating, through suitable axioms, the number 1 to true propositions and the number 0 to false propositions. One can also consider, besides classical logic, with two truth values, also multivalued logics, or, more generally, logics of various kinds, with their syntactical operations and their semantical interpretations.

Turning to topics more specific of mathematics, the following can be engrafted (as it has been done in [11]):
sets through the quality $Qins$ and the binary relation $Rins$;
collections through the quality $Qcoll$ and the binary relation $Rcoll$;
functions through the quality $Qfun$ and the univocal ternary relation $Rfun$;
correlations through the quality $Qcorr$ and the ternary relation $Rcorr$.

If we assume that $Qins$, $Qcoll$, $Qfun$, $Qcorr$ are metaqualities, and that $Rins$, $Rcoll$, $Rfun$, $Rcorr$ are metarelations, we obtain a good context for engrafting various theories of sets, starting with Zermelo-Frænkel set theory and Gödel-Bernays-Von Neumann class theory.

From set theories one can pass to the study of the connections between sets and natural numbers: both "neopythagorean" and "non-pythagorean" approaches are feasible. The first important decision to be taken by whom is choosing a "neopythagorean" approach is that of assuming that $Qnnat$ is a metaquality and that $Rnsuc$ is a metarelation. This decision is indeed necessary in order to take the "true" natural numbers as a manageable basis of mathematics, logic, and computer science.

From arithmetic one can pass to analysis, constructing rational and real numbers in the usual ways. More interesting could be, perhaps, to pass directly from the general theory of sets and functions to the engrafting of *general real*

numbers, comprehending standard and nonstandard reals, as objects of a *qualitatively new type*. To this aim one could introduce the quality $Qgreal$ of being a general real number and the binary operations $Gradd$ (*addition* of two general real numbers), and $Grmult$ (*multiplication* of two general real numbers). One could introduce thereafter various models of the analysis (standard and nonstandard): to each model $\mathcal{M}$ would be associated a set $\mathbf{R}^{\mathcal{M}}$, the *set of real numbers of* $\mathcal{M}$, whose elements should always enjoy $Qgreal$.

Besides standard and nonstandard analysis, classical and non-classical logics, many more branches of mathematics, logic, and computer science could be engrafted, introducing variables, categories, algorithms, probability, information, etc.. The boldest persons might try and engraft branches of other disciplines, introducing, for instance, phyisical, biological, economic laws, theories and experiments confirming or belying them, ethical or juridical principles, natural and artificial languages, etc..

Programming the exploration of the various, rich, colourful world that should enlarge "Cantor's paradise" is a difficult task. One can only say that the success of the exploration will probably depend on the number and variety of the "explorers", on their ability of appreciating the best traditions and the greatest cultural achievements of the past, in conjunction with the faculty of imagining which innovations might be most fruitful and valuable. It seems utmost necessary to combine imagination and scientific rigor, intending rigor in the broader sense outlined in [2000], where we assert:

> Any investigation of the fundamental axioms of mathematics, logic and computer science, as well as of various experimental, human and philosophical sciences needs, among other things, to overcome a too restricted vision of the different specialities, and requires a broader idea of mathematical and scientific rigor. Mathematical rigor is not only carefulness of the proofs, but also engagement in exposing, in the most clear and understandable way, the problems one would want to solve, the theorems one would want to prove, the conjectures one would want to verify or refute. We think that scientific rigor ultimately consists in clearly and frankly exposing their own certainties and doubts, which problems one believes to have solved and which one would like to solve or see solved, while avoiding those confuse, obscure, uselessly complicated talks that end up in annoying even the most favourably disposed listener.
>
> Summing up, we may conclude that any consideration on method, rigor, and meaning of science leads us in the end to the ancient intuitions of the sapiential value of humility, of "conviviality" (which keeps together sharing of knowledge, friendship, search of mutual understanding), and of trust in Wisdom, which meets all those who love and seek it.

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